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ELEMENTS OF MATHEMATICAL EPISTEMOLOGY – ELEMENTS OF THE PHILOSOPHY OF TEACHING MATHEMATICS

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Abstract

In the paper we present the main idea of the concept which we have called confrontational concept of mathematical epistemology. We refer it to philosophy of mathematics (in the context of epistemology of research) as well as to didactic problems (in the context of teacher preparation). Although we tried not to involve our discussion directly with any existing concepts of the philosophy of mathematics, however, in the paper one can notice some elements of modern formalism as well as Lakatos quasi-empiricism or a modern approach to structuralism.

1. Introduction

Any philosophy of mathematics has powerful implications for social and educational issues and many didactic consequences.

P. Ernest

Let us start first by considering the idea of E. P. Wigner, described by R. Murawski in the introduction to the article *Inconceivable effectiveness of mathematics in the natural sciences* ([13]):

Wigner states that mathematics has no content and it is only a "formal fun". A mathematician therefore does not have any real knowledge, but only some specific skills of deft concept manipulation. A prime example, quoted by the author, are complex numbers. They have no analogue in the

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real world. They are only a very useful tool. They have no ontological background.

If we are surprised by such an attitude to mathematics, then we unfortunately must swallow a bitter pill – it is a very popular point of view, even among highly educated people. Why? Because they have encountered such mathematics at school, college and university, even in books that aim to popularize maths. Such mathematics is given for applications to engineers, physicists, chemists and... future maths teachers.

Is it dangerous? Definitely, YES. It leads to the opinion that it is impossible or even unnecessary to understand mathematics: that it is sufficient to know which formulas are necessary in a particular situation and apply them. Those formulas are useful, but only by accident... A hammer is also useful, but frequently one can manage without it replacing it, for example, with a stone... Why do such opinions arise? One cannot escape these opinions even when formulas usefulness is demonstrated or when it is shown that they are the best methods for solving a problem. They are caused by a lack of understanding of "the spirit of mathematics" and a lack of basic reference to the philosophy of mathematics.

The aim of this paper is not to present detailed characterizations of different concepts within the scope of the philosophy of mathematics. We completely omit, for example, ontological problems. One can find detailed information about historical and modern approaches to particular branches in books e.g. [11], [12], [13]. More information, in particular referring to intuitionism and formalism can be found in other sources as well as on Internet. Presenting them or referring such literature related to these theories here would be an unnecessary lengthening of this article. Our aim is to show a new concept which although rooted in old ideas of the philosophy of maths, will shed new light on contemporary problems of mathematical epistemology.

2. Epistemological basis and the concept of confrontation

Let us start by specifying the meaning of notions which will be used in the article.

By a *mathematical theory* we will mean a theory built on a specified axiomatic system, preserving inference rules. We will not touch on logicism[1]

[1]More contemporary approach to logicism can be found in the studies of H. Weyl ([21], [22]) and E. T. Bell ([1]). Currently, this concept has been rather absorbed by the formalism.

and formalism too much here[2]. Thus analysis of formal theories is not essential in our considerations. Roughly speaking: Mathematical theory is the (mathematical) universe in which we currently operate. We can change or modify it (for example by adding or eliminating axioms), but we "function" within it, and our activities have to be focused on discovering it as effectively as possible.

A *mathematical structure* is a set of known mathematical objects (definitions, theorems, local assumptions, examples etc.) creating a connected system within the scope of a specified mathematical theory. It is a very broad notion. We can consider, for example, topological structures, but we can also distinguish some specific structures: the structure of general topology or the structure of metric topology (the second one, is in a natural way, contained in the first one). One can also talk (using mathematical language) about, for example, structure of algebraic topology. Sometimes it is difficult to distinguish and name a mathematical structure.

A system of specified theory and the mathematical structures complemented by the genesis (history) of a problem[3] will be called an **epistemological basis**.

Let us consider some examples.

The first one is connected with the notion of **entropy** in discrete dynamical systems.

Let us start with a mathematical theory. Obviously, the basis of all analysis contains, for example, the axioms of the set theory, although it is not always emphasized by researchers. However, if "continuum hypothesis" is employed, then reference to axiomatic basis is essential[4].

Yet, this example very consciously takes into account the history of the problem and the *mathematical structure*, in which considerations are carried out.

Now we will quote (with slight editorial modifications) a fragment of the article by E. Korczak-Kubiak, A. Loranty, R. J. Pawlak ([10]):

First, we present some intuitive description of problems connected with information system and information flow. Assume that we have a set X of elements (information) divided into a finite number of disjoint subsets $\{A_1, A_2, \ldots, A_k\}$, which are distinguished on the basis of fixed attributes (this partition is denoted by P). Suppose also that we have a probability

[2]We do not touch on means: we do not create derivative interpretations even in such contemporary works as [16].

[3]It is worth noting here the relationship of epistemological basis with Principle of parallelism mentioned, among others, by R. Duda ([6]).

[4]In the context of this issue in relation to discrete dynamical systems, prof. P. Szuca drew my attention to the work of A. Blass ([2]).

measure μ on X, so $\sum_{i=1}^{k} \mu(A_i) = 1$. Then we may assign to the partition P the number (the entropy of partition) defined in the following way:

$$H(P) := -\sum_{i=1}^{k} \mu(A_i) \cdot \log \mu(A_i).$$

Roughly speaking, if partition P describes a state of information flow, the number $H(P)$ may be regarded as a "measure of uncertainty". If $H(P) = 0$, then situation is defined precisely – measure is focused on some set A_{i_0} from the partition P (i.e. $\mu(A_{i_0}) = 1$). Moreover, we can say that the higher the entropy of partition is, the greater uncertainty is (in this case, the measure is more evenly distributed over the different sets of the partition).

After a given period of time, elements of X change the values of their attributes and thereby they "move" to the other sets. Perhaps a new partition of X (onto sets measurable with respect to μ) is created. These changes are described by a certain (invariant) function – let us denote it by ϕ. After the next unit of time, the elements "move" again and we obtain a new partition of X. The changes are described by the function ϕ. It means that in comparison to initial state these changes are described by the function $\phi^2 = \phi \circ \phi$. Going further in this way, we obtain the dynamics of the function ϕ. The entropy of ϕ with respect to the measure μ is defined in the following way. Let $\mathcal{P} = \{A_i : i = 1, ..., m\}$ be a decomposition of X such that A_i are μ-measurable for $i \in \{1, ..., m\}$. The metric entropy of ϕ with respect to the measure μ is the number $h_\mu(\phi) = \sup_{\mathcal{P}} h_\mu(\phi, \mathcal{P})$, where

$$h_\mu(\phi, \mathcal{P}) = -\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{B \in R_{n-1}(\mathcal{P})} \mu(B) \cdot \log \mu(B)$$

and $R_{n-1}(\mathcal{P})$ is the set containing all intersections of the form

$$A_{i_1} \cap \phi^{-1}(A_{i_2}) \cap ... \cap \phi^{-(n-1)}(A_{i_n}).$$

The entropy of this function determines the level of uncertainty of dynamics of function ϕ. If it is 0, then we can talk about a certain stability of this dynamics. If it is greater than 0, we can say that this dynamics is chaotic and the number qualified as the entropy can be considered as a certain kind of "measure of chaos".

An important way of "describing chaos" of certain actions directly related to the structures that could be conventionally called "metrical" was created in this manner. This, however, was insufficient. Many considerations are based on topological structures, so that these achievements were *confronted* with them.

In the sixties of the twentieth century R. L. Adler, A. G. Konheim and M. H. McAndrew introduced the notion of the topological entropy of a continuous function $f : X \to X$ defined on a compact space X.

However, introduction of this notion could not be isolated from its historical context, so that there were attempts of *confronting* it with metrical approach. In 1971 T. Goodman proved the variational principle determining the relationship between the topological entropy and the entropy with respect to measure, which in the most general form, taking into account later research, one can write in the following way:

Theorem 1. *For any function f we have:*

$$h(f) = \sup\{h_\mu(f) : \mu \textit{ is a probability } f\textit{-invariant Borel measure on } X\}.$$

One should mention in passing that the changes mentioned above, were also the consequence of *confrontation* with different structures. The starting points were structures based on continuous functions, and later analogues were searched for, in the case of less obvious structures of Darboux-like functions: [19], [4]. Simultaneously, studies have been carried out, where existing structures connected with examinations of discrete dynamical systems are *confronted* with algebraic structures (see [7] or [8]) or with generalized topological spaces (see e.g. [15]).

It is worth noting here the emergence of the generalized topological spaces.

In many issues, the assumptions associated with topological spaces are very difficult to achieve. Indeed, suppose that we have a finite or infinite set X and a dynamics on the elements of this set described by a function or multifunction T. To be able to apply mathematical tools, we need some (topological) structure on this set. How to define it? If we examine the dynamics of the function, it is natural to distinguish such sets A for which $A \subset T(A)$ (in the notation which is typical for the theory of discrete dynamical systems: $A \underset{T}{\to} A$). Thus it would be convenient to consider such a family of sets, but it will not be a topology. We can *confront* this situation with generalized topological spaces introduced by Á. Császár ([5]).

The problems we have mentioned need not be referred only to new notions, but also to the properties of existing objects. Brouwer Theorem on a fixed point does not lead to new notions (in opposite to Banach Theorem which gives the possibility to distinguish a new class of functions: contraction). Simultaneously, it is frequently implied that new notions necessitate expansion of the existing structures. Then we return to the logic (and fixed inference rules), understood as the specified mathematical theory. However, sometimes we need to modify our theory. In particular, note that our problem may be unprovable within the theory (Gödel discoveries). A meaningful example of such a situation is the continuum hypothesis. Obviously, then we can add a new axiom or simply assume it locally. It is in accordance with Russell concept (following R. Murawski [13]):

Speaking about the difficulties faced by the doctrine of logicism, Carnap first mentioned of problems related to the fact that in the proofs of many mathematical theorems it is necessary to use certain principles that are not purely logical, for example, the axiom of infinity and the axiom of choice. Russell, however, has found a simple solution to this problem i.e. considering in this case not the theorem T, but the implication $A \Rightarrow T$, where A is, for example, the axiom of infinity. By the theorem of deduction, this implication is then a logical theorem.

In order to illustrate this problem let us present a piece of research led, among others, by prof. T. Natkaniec (see [14]).

Problem. *Is every function $f \in D^*(\mathbb{R}, \mathbb{R})$ (i.e. function having a dense level) a composition of (two) almost continuous functions?*

Let us adopt the following denotations:

$A(c)$ – the union of less than 2^ω many first category subsets of $\mathbb{R}$ is of the first category again.

$A(m)$ – the union of less than 2^ω many Lebesgue measure zero subsets of $\mathbb{R}$ is of the Lebesgue measure zero again.

It is well known that these two conditions follow from Martin's axiom and also from Continuum Hypothesis.

A general answer to the problem of ZFC, presented above has not yet been found. Nevertheless, it is possible to give some answers that "enrich" the theory (speaking less scientifically: by introducing some new assumptions to our mathematical universe).

Theorem 2. ([14]). *Assume $A(c)$. Then each $f \in D^*(\mathbb{R}, \mathbb{R})$ can be expressed as a composition of two almost continuous functions.*

It is clear that we have here an application of Russell's concept. The last theorem is a certain solution to this problem. Is it the ultimate one? Time will tell. For now, the solution has been included into the structures of real analysis and topology.

There is one more important element which should be emphasized: joining the structures. A proper example here is famous proof of Fermat Theorem[5]. Sometimes we definitely have to go beyond our structure.

At the fringe of our considerations we should mention that during the necessary *confrontation* within the scope of a mathematical theory as well

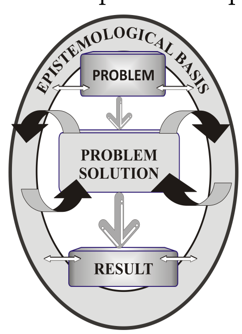
[5] An interesting description of the problem as well as a sketch of solution connected with elliptic curves can be found for example in [20].

as mathematical structures, one can meet epistemological obstacles[6] that may make the process of discovering mathematical facts slower.

What is the main idea of the concept we have called the **confrontational concept of mathematical epistemology**?

First we have the PROBLEM, which sometimes results from the *confrontations* of existing solutions, and of existing necessities with respect to different epistemological basis – for example problem of entropy with respect to measure and topological entropy. The basic element is *confrontation* of the problem with an epistemological basis. Further, the process of SOLVING THE PROBLEM begins. Here, we can refer to rich literature within the scope of the philosophy of mathematics, as well as didactics of mathematics. The solution to the problem appears as a result of continuous *confrontation* between intuition and formalism, supported by motivating factors, with continuous *confrontation* of obtained (partial) results with epistemological basis (such idea is wider than formalistic approach). Finally, THE RESULT appears. We avoid using the phrase “solution to the problem” intentionally, as it is frequently only a partial solution or statement that the problem is unsolvable. The result is also *confronted* with epistemological basis.

The graphical illustration of the process is presented below.



Although we have declared we should not touch on logicism and formalism, it is clear that in the concept described above one can easily find elements of modern formalism[7] as well as Lakatos quasi-empiricism[8] and a modern approach to structuralism[9]. However, our concept does not contain the whole scope of the philosophy of mathematics. It is, in a sense,

[6]This notion is due to G. Brusseau (see [17]).

[7]A particular role is played here by ZFC axiom system mentioned earlier. Moreover, iterative concepts and problem of infinity seems to be interesting.

[8]The wider context of quasi-empiricism would arise in the analysis of the process of solving problem. Particular importance in the context of these issues would be critical rationalism and rational falsification.

[9]Studies of C. H. Parsons are particularly interesting here.

its power. For example, we do not analyse whether the infinity should be treated only hypothetically and we do not analyse the sense of existence of infinity axiom. If we examine an issue where we need such assumptions, then we *confront* these needs with existing knowledge of them. We check in which structures the analysed problems exist and how we can or should change the scope of these structures. Finally, we are interested in the history of the problem: how to cope with this particular or similar problem(s).

3. Didactic issues

> **In the framework of the range of important topics in the theory-practice relation, I shall concentrate on certain aspects of the mathematical knowledge negotiated and mediated in this relationship. Theoretical perspective will not be curricular, historical, or mathematical, but an attempt to use the epistemological basis[10] of mathematics. If it is accepted that epistemology is the scientific enterprise of investigating the status, structure, and meaning of knowledge, then this perspective becomes indispensable for the analysis of such indirect modes of cooperation between scientific didactics and everyday teaching practice that aim a communication as a reciprocal dialogue searching for possibilities off constructing and enhancing meaning and not simply conveying knowledge matter. The intention is not to describe the mediation of a coherent didactical theory named "mathematical epistemology" to the practice of mathematics teaching, but to stress and to use epistemological considerations of mathematical knowledge, because this is an essential characteristic of every process of mediating knowledge between teacher and students as well as between researcher and teacher.**
>
> **H. Steinbring** ([18])

Mathematical education is a complex issue which is examined and analysed in various ways. It is no wonder that within the scope of didactics of mathematics there are many papers whose parts deal with mathematical philosophy and the philosophy of didactics of mathematics (e.g. articles by P. Ernest, M. Otte and F. Seeger, M. Niss, or R. Noss contained in monograph edited by: R. Biehler, R. W. Scolz, R. Strässer, B. Winkelmann Didactics of mathematics as a scientific discipline, Kluwer Academic

[10]The meaning of the notion "epistemological basis" in the H. Steinbring's article (1994) is a little different and narrower than in our paper. However this quotation may constitute some kind of introduction to this part of our article.

Publishers, Math. Education Library; and in Poland, in a very interesting article by E. Jagoda, M. Pytlak, E. Swoboda, S. Turnau, A. Urbańska ([9]).[11]

What do we mean by the **philosophy of didactics of mathematics**? First of all we should refer directly to mathematical epistemology, so that we understand it as a *science examining general cognitive processes within mathematics*. Of course we do not refer to the whole issue, but we concentrate on 'teaching philosophy' (in some coherence of learning, but also with a certain distance to the student, as referred to a teachers' approach). Before presenting our concept, we will formulate a general "axiom"[12] concerning this issue:

Nowadays, building mathematical knowledge of individuals must contain elements of confrontation with the modern educational and epistemological basis.

Leaving the above statement without any comment may cause a lot of misunderstandings.

Let us start with a simple statement: The above axiom refers mainly to a teacher[13]. It concerns a student only indirectly: through the impact of a teacher (not through requirements!). Those involved in the process of acquiring knowledge are "getting used to" *confronting* their activities with a contemporary educational and epistemological basis (within the scope of the availability of these activities).

It is also important to join modern educational and epistemological basis by a conjunction. An essential problem for a teacher is the capability to include innovative didactic and educational solutions into the teaching process, and *confronting* the effects with an epistemological basis. It is directly connected with the following observation:

If a teacher does not have any knowledge of or reflections on the philosophy of mathematics, he or she will "create" his own philosophy, often imperfect and highly confined, and sometimes burdened with many errors. His/her attitude and seemingly unimportant comments will create in students' minds a completely false view of mathematics – a view which often turns into a reluctance to pursue the subject. Let us look at some examples.

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[11]Of course, these are only examples. In fact, it would be possible to mention more authors here.

[12]The use of the word "axiom" is to show the relationship with the general concept of Maths. One can use words like: principle, suggestion, etc., but in our opinion it would not show the heart of the matter. This thesis is set to be an axiom (in the absolute understanding of this word).

[13]We mean here also someone preparing e-learning materials.

Negative numbers.

Let us start with the historical background. Negative numbers were introduced into mathematics quite late. The Babylonians did not know them (although they certainly knew the concept of debt). The Chinese in II century BC made some observations connected with them, and in Greece only Diofantos from Alexandria (III century) used some of their properties. If we follow more closely the history of these numbers, we will see that practical motivation was not sufficient to develop their wider theory. The introduction of these numbers to mathematics is due to research by Descartes and I. Newton. Modern approach to arithmetic of integers appeared in XIX century.

By the didactic principle of parallelism we already have the first *confrontations* with the epistemological basis (history of the problem). Let us repeat: the notion of a debt was not enough motivation to consider such numbers! When was the breakthrough? The first step forward came as a natural need "to extend the axis" and from Descartes' work. This is the first observation.

Now, let us consider further *confrontations*: one of the basic constructions of these numbers in the modern theoretical arithmetic is the method using equivalence classes of pairs of positive integers. Thus we have our next observation: a negative number should be related to an (ordered) pair of numbers.

Let us perform one more *confrontation*. Pairs of numbers related to integers should be connected with operations on these pairs. What is more, these operations have to agree with the basis of ring structure and they can not "change results of operations for positive integers".

All these things constitute the <u>knowledge of a teacher</u>, which should be used while developing definitive didactic solutions aiming to introduce these numbers.

How do we translate this knowledge into examples of a methodical solution? We will be avoiding detailed solutions here. We will only present a general scheme.

Students know the number line (semi-number line) "with beginning at 0". We draw the number line and try to interpret operations like $2 + 3$, $8 - 2$, $6 - 4$, etc. with respect to specific issues such as, for example, inflow and outflow of money. In this way we realize a request of one of *confrontations*: we operate with pairs of numbers. At some point, we meet the operation $2 - 3$. The point is not to give a result but to interpret the operation on the number line (corollary from the principle of parallelism). May be it would be advantageous to give such a problem to students several times. At some point a natural solution will appear: "extension of the number line into the left". In such a way, new numbers will appear. How should they be denoted? Again, it is worth to *confront* it with the epistemological basis in the interpretation of our needs. Numbers obtained from operations $2-3$ and $0-1$ are the same. Similarly the same are the numbers derived from operations $3-5$ and $0-2$. We could therefore denote these numbers by $0-1$, $0-2$, ... As 0 would always repeat, we can skip it and we obtain -1, -2, ... And now the problem of operations. Here we come to the heart of the *confrontational method*! We should not create the conviction that we deal with the following situation: the operations are "somehow" defined and "accidentally" they have the same properties as they have for positive integers.

We definitely need to emphasize that the operations should be defined in such a way to have respective properties (associativity, commutativeness, a neutral element, etc.)

We do not have to name these properties. We can discover the computation ways with students, but we should "convince them" that this is not accidental, that the power of mathematics lies in its structures. Definitely, the matter is not learning structures, or even through the structures, but within the *confrontation* to these structures[14].

Let us come back for a moment to the idea of E. P. Wigner (at the beginning of the paper). The power of the application of complex numbers does not come from the fact that someone has discovered, during sleepless nights, that we can solve the equation $x^2+1=0$. Then, it would be enough simply to join a symbol to the set of real numbers and to make the agreement that this is a solution to this equation. In fact, it would not give us anything because, for example, the equation $x^2+2=0$ would still not have a solution. "The power of complex numbers" lies in the fact that the considerations for that problem have always been connected with certain structures, and what is more, the isomorphism of Gauss and Hamilton solutions was obtained.

Similar solutions can be shown with respect to a derivative of a function. If you do not *confront* didactic solutions with an epistemological basis, then the notion will be very difficult. Its usefulness will seem completely accidental, and it may discourage rather than motivate the student to work. The sources of the concept of derivative can be found in physics. So let us start with the simplest problem[15], for example:

Within the first five seconds of a motion, the dependence of distance on the duration x is described by the function $f(x)=5x^2$.

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[14]Meaningful examples here are quaternions and Cayley octaves. We are forced to restrict the structures here (for example, multiplication is not commutative).

[15]Obviously one can relate the problem for example to falling objects. The presented function is "close" to the description of a free fall.

Now, we give a free hand to students, simultaneously directing them to the question: how was the object velocity, for example, in the third second of the motion? The problem is that we do not even know how to say: what is a velocity at a moment? However, we can compute the average velocities in successive intervals with an excess (between the 3-rd and the 4-th second) and with insufficiency (between the 2-nd and the 3-rd second). Thus the object velocity at the 3-rd second of the motion is within the range of 25 m/s and 35 m/s. By decreasing the intervals (for example $(3; 3.5)$ and $(2.5; 3)$) we obtain more accurate estimations: the range between 27.5 m/s and 32.5 m/s. Obviously one can further decrease the intervals (calculations using a calculator or computer), but very soon one can find that the obtained ranges "will converge" (whatever it means) to the value of 30 m/s. This is a derivative $f'(3) = 30$. Now we must consciously *confront* our discovery with an epistemological basis. As a result of this *confrontation*, the average velocity will change into the difference quotient and converging of the obtained ranges into the limit of difference quotient. It may also become possible to find an answer to the question: what does instantaneous velocity mean?

We have presented the examples of *confrontational method* with respect to introducing new notions, because it is particularly easily seen in such a case.

We should once more emphasize: the presented "axiom" is related mainly to a teacher and to his/her didactic and methodical choices. Just as the mathematical maturity of students grows, the style of *confrontation with epistemological basis* should be included into their direct actions.

In conclusion, it should be emphasized that if we do not apply this "axiom", it may lead not only to a lack of understanding of mathematics as a science, but also to a lack of understanding of the mathematical tools we use. The classic example is the ancient method of analysis (analisis antiquorum) for solving equations.

4. Conclusion

We are aware of the fact that the article has an introductory character, in the sense of "inducing the discussion". Mathematics is surprisingly useful. To fully understand why the moth flying into to a bulb, continues to approach it until its death, you need to know the logarithmic spiral (Bernoulli)[16]. In order to investigate certain sociological or economic issues we apply a Gaussian normal distribution using the number π, whose genesis has nothing to do with these issues. The solutions of logistical problems are often based on difference and differential equations, which arose

[16]Following K. Ciesielski, K. Pogoda [3]).

when the scale of the logistical problems did not require too sophisticated mathematical tools. Finally, climate change, some aspects of biology and environmental protection, or even elements of art, are examined and analysed using mathematical rules that were created without any connection with these applications.

It all indicates that in the modern world mathematics is needed more than ever before. Many mathematics educators indicate that one of the most important tasks of modern teaching of mathematics is the problem of the relationship of theory and practice[17]. This is a very important observation! However, a one-sided approach to this issue may in the long term bring about the opposite effect. Simple mathematical issues may be applied for simple problems only. This is needed but at school level we can not go further. However, to understand many of the issues "the modern world" needs higher mathematics, what's more, using mathematics, not previously met. One of the students of doctoral studies talked about her practice at the Ministry of Foreign Affairs. She had to perform analysis using difference equations, which she did not know. Moreover, she could not apply directly the found facts – they had to be modified. That is the example of situation people in modern world must be prepared for. In our opinion, one of the elements of such preparation will be awareness of the need to *confront* one's thoughts with epistemological basis. Such an awareness must be built (gradually and systematically) and maintained by a teacher, but first a teacher must have knowledge in this field and be convinced of such a need.

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IS MATH A GIFT?

BARBORA BARTOŠOVÁ

Abstract

"I am just not a math person" is a sentence we can hear very often these days. Students use it as an excuse any-time they do not know or understand anything. But is Maths really some gift a person gets or is not? Or is it more like ability which can be improved if practised enough? This is a brief preview of what do the Czech, Slovak and also foreign authors think of this issue.

1. Introduction

"He/she is a real talent, math genius!" and similar sentences are often said by teachers, students and parents. These shift the view of mathematics as something from wonderland which ordinary people can not understand. They say "I have never been good at math, I don't have the gift for that". But is it the truth?

2. Theoretical Base

Ability is commonly understood as the set of mental qualities of an individual which are essential for successful accomplishment of specific kind of activity. According to this, in case of mathematics, the greatest ability has the pupil who wins the Mathematical Olympiad. Other understanding of ability is that it is the prerequisite of (fast) development and improvement in certain field. According to this view the ablest student is not the one who performs better but the one who, under the same conditions, evolves more. Ability is quite consistent quality, it is changing slowly. *Mathematical abilities* are defined in many ways. It is necessary to distinguish these basic factors:

- numerical factor – manipulating with numbers, fast and precise calculations;

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- spatial factor – important in geometry as well as in arithmetic (e.g. layout of space while writing calculations, positional number notation etc.);
- verbal factor – solving of verbally formulated problems;
- reasoning factor;
- general intelligence.

Skill is ability specialized for specific use, set of tools an individual adopted by learning or practice and which are essential for performing of specific activity. But what is important – it is not automatized and stereotyped. Skills improve with every use so every time it is something new to a certain extent.

In Mathematics more than in any other subject, the quality of performance is dependent on systematic gaining of special skills for understanding and solving certain types of problems. Such skill for pre-school children may be the skill to count from one to ten. Especially young children learn these skills spontaneously. Children at schools learn skills in reading and writing numbers, performing various arithmetical operations, managing and using of multiplication table, handling decimal numbers and fractions, operating with powers and roots etc. In geometry they adopt drawing of different geometrical figures, handling measures and weights, applying different formulas and theorems to geometrical problems etc. It is then understandable that children with high level of mathematical abilities who are neglected might have insufficiently developed and fixed *Mathematical skills*, or knowledge.

Talent is defined as inborn anatomical-physiological disposition of an individual, which is prerequisite for ability.

Mathematical talent is genetically determined. This is confirmed for example by the existence of very mathematically talented families, e.g. Bernoulli. The other confirmation were the researches with twins, which show great similarity between young children but as they grow older, the similarity is not so significant. That might be caused by the fact that for older children there are more problems in tests which require special skills or learning vast amount of school knowledge of mathematics. Which is why for younger children the genetic congeniality shows convincingly.

3. Study

Carol S. Dweck from Department of Psychology at Stanford University led the research asking the question: *Why aren't there more women in math and science?* It shows that there is a difference in how students cope with experiences that may question their ability – they can feel challenged or demoralized by them.

At the beginning there are students' beliefs about intellectual abilities. Do they view it as a gift, something you simply have or you don't or is it something that can be developed through practice and dedication for them? Former researches show that viewing ability as a gift led students to question their ability and lose motivation when they encountered setbacks. This affects more females than males. Moreover it is shown that these females are more susceptible to stereotypes which was confirmed by following female students at Columbia University through their calculus course. Many students thought that stereotyping was alive and well in their calculus section. It had a strong negative impact on women with belief of math ability as a gift in contrast to those who viewed it as something they could augment (it had only little impact on these women.)

So how can these types of women be encouraged? Obvious solution is to look for opportunities to praise these female's ability. But this approach is wrong because there are many undesirable consequences. It tells them that their ability is a gift. When these students hit a period of difficulty they tend to lose their confidence. As a result they lose interest in pursuing the task.

Much more better solution is addressing their beliefs about the nature of ability. There are some studies showing that if you convince students that intellectual skills could be acquired rather than simply bestowed as a gift it leads to important gains in females' math achievement. But can we use this in our own math classes? Of course we can. It is shown that even those subtitle "innocent" information about "great or genius mathematicians" used only to make the lesson more interesting influence students negatively. We should focus on presenting these persons as people who were deeply interested in and committed to math, and who worked very hard. This little difference can change our students' lives.

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SOME REMARKS ON FLUID FLOW IN HOURGLASSES

ARKADIUSZ BRYLL, ROBERT SOCHACKI

Abstract

In the paper the authors analyse different shapes of an hourglass for the linearity of their graduation. We also assume that any hourglass (more precisely, each of the two congruent parts) has the shape of a solid of revolution and any cross section at height h of this hourglass depends on the base radius r, i.e. $h = f(r)$.

1. Introduction

In the paper [2] the author analyses three different shapes of an hourglass for the linearity of their graduation. We think (and this is the purpose of this paper) that it is worthwhile expanding and generalizing the results contained in the above mentioned work. Accordingly, we will use the simplest knowledge of higher mathematics (see [1]). More specifically, we will make use of the solution of a differential equation with separated variables in the form:

$$F(h)dh = \alpha dt,$$

where $F(h)$ is a given function and $\alpha = const$. We also assume that an hourglass (more precisely, each of the two congruent parts) has the shape of a solid of revolution and any cross section at height h of this hourglass depends on the base radius r, i.e., $h = f(r)$. The flow of liquid from the upper part to the lower part depends on the shape of an hourglass, and the hole cross section of the flow.

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2. Main results

We will consider hourglasses for which the upper part is described by the equations of curves: $r = r_0$, $h = ar^b$, $\frac{r^2}{a^2} + \frac{h^2}{b^2} = 1$, $\frac{r^2}{a^2} - \frac{h^2}{b^2} = 1$, $(a > 0, b > 0)$. We rotate these curves around the axis Oh in the co-ordinate system Orh. Moreover, we will analyse an hourglass in which any change of the liquid column depending on the time has a linear dependence. Rotating the mentioned above curves, we obtain the following solids of revolution: a cylindrical surface, a conical surface (when $h = ar$, $a > 0$), a paraboloid of revolution, and a hyperboloid of one sheet (see Fig. 1).

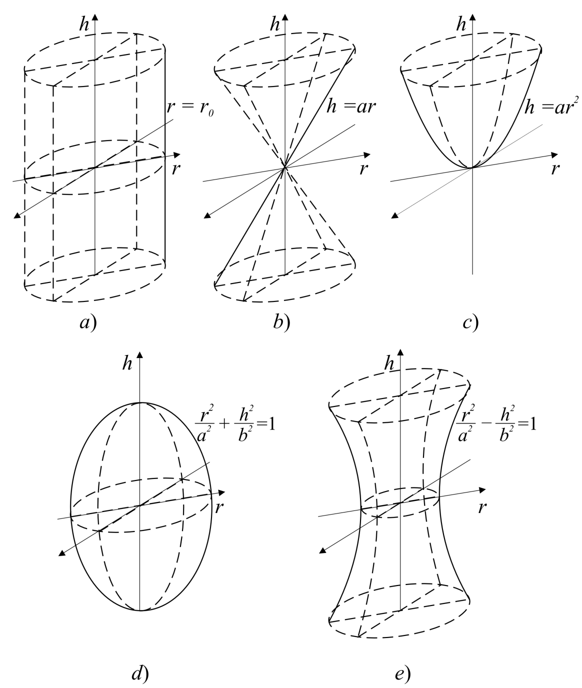

Figure 1

Let $r(t)$ be the base radius of the cross section of an hourglass at height $h(t)$ in the time t (see Fig.2).

The volume of the liquid passing through the cross-sectional area of S at the time t has the form:

$$V(t) = v(t) \cdot S \cdot dt, \tag{1}$$

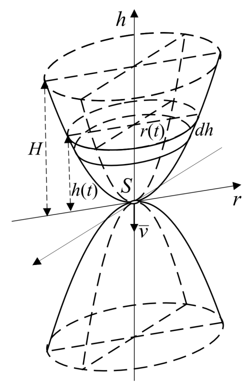

FIGURE 2

where $v(t)$ is the flow velocity of the liquid through the cross-sectional area of S. By the Bernoulli law, the velocity is:

$$v(t) = \sqrt{2g \cdot h(t)},$$

where g is the gravitational acceleration. On the other hand, we have the following formula:

$$V(t) = -\pi r^2(t)dh, \qquad (dh < 0). \tag{2}$$

Using (1) and (2), we obtain:

$$-\pi r^2(t)dh = v(t)Sdt,$$

i.e.

$$-\pi r^2(t)dh = \sqrt{2g \cdot h(t)}Sdt.$$

Hence:

$$\frac{r^2(t)dh}{\sqrt{h(t)}} = -kdt, \tag{3}$$

where $k = \frac{\sqrt{2g}S}{\pi}$. Thus, we obtain the differential equation with separated variables. The solution of this equation depends on the choice of the radius $r(t)$. We will consider four cases.

I. $\quad r(t) = r_0, \quad r_0 = const$

If $r(t) = r_0$, then the solid of revolution has the shape of a cylinder (see Fig. 1a). Then, the formula (3) has the form:

$$\frac{dh}{\sqrt{h(t)}} = -\frac{k}{r_0^2}dt.$$

Thus, after the integration we obtain:

$$2\sqrt{h} = -\frac{k}{r_0^2}t + C.$$

Based on the assumption that $h(0) = H$, the integration constant C equals $2\sqrt{H}$. Thus, we have:

$$2\sqrt{h} = -\frac{k}{r_0^2}t + 2\sqrt{H},$$

$$h(t) = \left(-\frac{k}{2r_0^2}t + \sqrt{H}\right)^2,$$

hence:

$$h(t) = \left(-\frac{\sqrt{2g}S}{2\pi r_0^2}t + \sqrt{H}\right)^2. \tag{4}$$

If $h(t) = 0$, then the upper vessel is empty and all the liquid is in the lower vessel. Then, the total time of the flow of the liquid from the vessel top to the bottom one is:

$$t = \frac{2\pi r_0^2\sqrt{H}}{\sqrt{2g}S} = \sqrt{\frac{2H}{g}} \cdot \frac{\pi r_0^2}{S}.$$

Thus, the lower vessel of the hourglass can be scaled by the relationship (see Fig. 3):

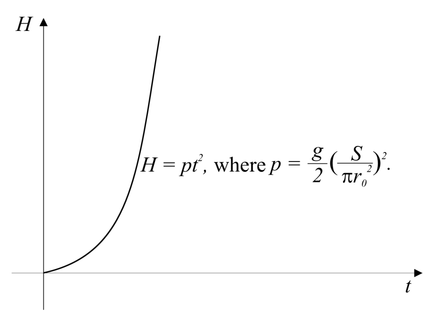

FIGURE 3

$$H = \frac{g}{2}\left(\frac{S}{\pi r_0^2}\right)^2 t^2. \tag{5}$$

II. $h = ar^b, \quad (a > 0, \quad b > 0)$

If $b = 1$, then the solid of revolution has the shape of a conic (see Fig. 1b). In the case of $b = 2$, the solid of revolution has the shape of a paraboloid (see Fig. 1c). We will solve the equation (3) for any a and b ($a > 0, b > 0$). From the assumption $h = ar^b$ it follows that $r = \left(\frac{h}{a}\right)^{\frac{1}{b}}$ and the equation (3) has the form:

$$\frac{\left(\frac{h}{a}\right)^{\frac{2}{b}} dh}{\sqrt{h}} = -kt,$$

that is:

$$h^{\frac{4-b}{2b}} dh = -ka^{\frac{2}{b}} dt.$$

Thus, after the integration we obtain:

$$\frac{2b}{4+b} h^{\frac{4+b}{2b}} = -ka^{\frac{2}{b}} t + C_1.$$

From the condition $h(0) = H$ we have:

$$C_1 = \frac{2b}{4+b} H^{\frac{4+b}{2b}}.$$

Thus, finally:

$$\frac{2b}{4+b} h^{\frac{4+b}{2b}} = -ka^{\frac{2}{b}} t + \frac{2b}{4+b} H^{\frac{4+b}{2b}}.$$

Hence we can find the formula of the function h:

$$h(t) = \left(-\frac{4+b}{2b} ka^{\frac{2}{b}} t + H^{\frac{4+b}{2b}}\right)^{\frac{2b}{4+b}}.$$

Taking into account that $k = \frac{\sqrt{2g}S}{\pi}$, we have:

$$h(t) = \left(-\frac{4+b}{2b} \frac{\sqrt{2g}}{\pi} Sa^{\frac{2}{b}} t + H^{\frac{4+b}{2b}}\right)^{\frac{2b}{4+b}}. \tag{6}$$

In the case of the conic ($r = \frac{h}{a}$, $b = 1$) we obtain:

$$h(t) = \left(-\frac{5}{2} \frac{\sqrt{2g}}{\pi} Sa^2 t + H^{\frac{5}{2}}\right)^{\frac{2}{5}}, \tag{7}$$

while, in the case of the paraboloid ($r = \left(\frac{h}{a}\right)^{\frac{1}{2}}$, $b = 2$) we obtain:

$$h(t) = \left(-\frac{3}{2} \frac{\sqrt{2g}}{\pi} Sat + H^{\frac{3}{2}}\right)^{\frac{2}{3}}. \tag{8}$$

On the basis of (6) we can determine the total time to empty the upper vessel. Taking $h(t) = 0$, we obtain:

$$t = \frac{2b\pi a^{-\frac{2}{b}}}{(4+b)\sqrt{2g}S} \cdot H^{\frac{4+b}{2b}}. \tag{9}$$

For the conic ($b = 1$), the total time to empty the upper vessel is:

$$t = \frac{2\pi a^{-2}}{5\sqrt{2g}S} \cdot H^{\frac{5}{2}}.$$

Since $a = \frac{h}{r} = \frac{H}{R}$, hence:

$$t = \frac{1}{5}\sqrt{\frac{2H}{g}} \cdot \frac{\pi R^2}{S}.$$

For the paraboloid of revolution ($b = 2$), the total time to empty the vessel is:

$$t = \frac{2}{3}\frac{\pi a^{-1}}{\sqrt{2g}S} \cdot H^{\frac{3}{2}}.$$

In the case of $a = \frac{h}{r^2} = \frac{H}{R^2}$ we obtain:

$$t = \frac{1}{3}\sqrt{\frac{2H}{g}} \cdot \frac{\pi R^2}{S}.$$

The lower vessel of the hourglass can be scaled by the formula (see (9)):

$$H = \left[\frac{4+b}{2b} \cdot \frac{\sqrt{2g}}{\pi} S a^{\frac{2}{b}}\right]^{\frac{2b}{4+b}} \cdot t^{\frac{2b}{4+b}}. \tag{10}$$

In the case of the conic ($b = 1$) we have:

$$H = \left[\frac{5}{2} \cdot \frac{\sqrt{2g}}{\pi} S a^2\right]^{\frac{2}{5}} \cdot t^{\frac{2}{5}}.$$

Moreover, if we consider the paraboloid ($b = 2$), then:

$$H = \left[\frac{3}{2} \cdot \frac{\sqrt{2g}}{\pi} S a\right]^{\frac{2}{3}} \cdot t^{\frac{2}{3}}.$$

III. $\frac{r^2}{a^2} \pm \frac{h^2}{b^2} = 1$

In this case the solid of revolution has the shape of ellipsoid (Fig.1d) or the shape of hyperboloid of one sheet (Fig.1e). Since $r^2 = \left(\frac{a}{b}\right)^2 (b^2 \mp h^2)$, thus the equation (3) has the form:

$$\frac{\left(\frac{a}{b}\right)^2 \left(b^2 \mp h^2\right)}{\sqrt{h}} dh = -k dt,$$

that is

$$\left(\frac{a}{b}\right)^2\left(b^2h^{-\frac{1}{2}}\mp h^{\frac{3}{2}}\right)dh=-kdt.$$

After the integration we obtain:

$$\left(\frac{a}{b}\right)^2\left(2b^2h^{\frac{1}{2}}\mp\frac{2}{5}h^{\frac{5}{2}}\right)=-kt+C_2. \tag{11}$$

From the condition $h(0)=H$ we can determine the constant C_2:

$$C_2=\left(\frac{a}{b}\right)^2\sqrt{H}\left(2b^2\mp\frac{2}{5}H^2\right).$$

Thus, finally, the equation (11) has the form:

$$\left(\frac{a}{b}\right)^2\left[\left(2b^2h^{\frac{1}{2}}\mp\frac{2}{5}h^{\frac{5}{2}}\right)-\left(2b^2H^{\frac{1}{2}}\mp\frac{2}{5}H^{\frac{5}{2}}\right)\right]=-kt, \tag{12}$$

where $k=\frac{\sqrt{2g}S}{\pi}$.

The upper vessel will be empty when $h(t)=0$. Then:

$$t=\frac{1}{k}\left(\frac{a}{b}\right)^2\sqrt{H}\left(2b^2\mp\frac{2}{5}H^2\right).$$

Note that the "$-$" sign refers to the ellipsoid, while the "$+$" sign refers to the hyperboloid of one sheet.

IV. $h(t)=-wt+H$

This case concerns the situation in which the change of the column of the liquid in the hourglass, depends linear on the time. Indeed, if $\frac{dh}{dt}=-w$ and $w=const$, then $h(t)=-wt+C_3$, wherein $h(0)=H$. So, we have $h(t)=-wt+H$ (see Fig. 4).

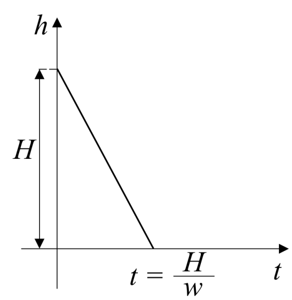

FIGURE 4

By the assumption $\frac{dh}{dt}=-w$, the equation (3) has the form:

$$\frac{r^2(t)}{\sqrt{h(t)}}(-w)=-k,$$

that is

$$\sqrt{h(t)} = \frac{r^2 w}{k},$$

hence

$$h = \left(\frac{w}{k}\right)^2 \cdot r^4. \tag{13}$$

This formula allows to determine the shape of the curve $h = f(r)$, if the height of the liquid column decreases linearly with the time and $f(r)$ is a bi-quadratic function.

3. Final remarks

As the above analysis shows, shaping hourglasses as typical solids of revolution does not guarantee the linearity of their graduation. This can be seen in the following equations: (4), (7), (8) and (12). If we still do want to obtain the linearity for the lower part of the hourglass, we must shape it another way. The shape is defined by the formula (13).

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GEOMETRIC TERMINOLOGY AND IMAGINATION

ROMAN GREBEŇ, JOSEF MOLNÁR

Abstract

How do pupils and students understand geometric terminology, symbols and phraseology? Using the same designation for a particular object, feature or link, allows the use of the same "language" to communicate about topics related to these concepts. The fact that everyone will equally understand the same term depends on how well the definitions are introduced. The objective of this article is one of the surveys, which is a part of the research, aimed to contribute to updating the geometric terminology in the Czech education system.

1. Introduction

Is it important to define and comprehend geometric terminology, symbolism and phraseology? Definitely YES!
The reason is obvious: Using the same designation for a particular object, feature or relation, allows the use of the same "language" to communicate about topics related to these concepts. Provided the definitions are well introduced everyone with equally understand the same term.

Historical frame. 25 years ago the crucial publications which were dealing with the Czech terminology of school mathematics were issued:

(1) Názvy a značky školské matematiky (*The terms and symbols of school mathematics*)

(2) Slovník školské matematiky (*The dictionary of school mathematics*)

The Terminology Commission for School Mathematics was established as a part of The Union of Czech Mathematicians and Physicists. The aim of this Commission is to update both publications and to carry on the efforts on the unification of school mathematics terminology. The focus of

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the research is closely related to this aim. The important part of it is the questionnaire survey examining geometrical imagination and knowledge of geometrical terms among primary school pupils, secondary school students and university students and mathematics teachers.

In February 2013 we initiated the research, which aims to find out what visions of basic geometric concepts primary students (of 6th – 9th grade), secondary school students (comprehensive schools as well as technical) and the school teachers of math have.

Research tools. Due to the character or the research, the most commonly used method will be questionnaire survey among those groups of respondents. They will give comments on various issues examining their images of geometric objects and relations between them, on the knowledge of definitions and understanding of the symbolic marking in geometry. The questionnaires focused on specific topic are used in the research. Respondents answer questions by indicating their preferred option and explaining or justifying their opinion. The survey is the third in the series.

2. Main results

2.1. Administration.

Questionnaire survey was performed in March and April 2014.

- Total respondents: 466
- Female/Male ratio: 1.3 (56% female, 44% male)
- Range of Age: 12 – 18 years

2.1.1. *Number of respondents: Role of respondent in the educational sphere.*

- **Elementary school, second stage** (6th – 9th grade; age from 12 to 15 years): 166 respondents
- **Grammar school, primary stage** (Eight-year grammar school; corresponds to elementary school, sec. Stage; age from 12 to 15 years): 128 respondents
- **Grammar school, second stage** (age from 15 to 18 years): 80 respondents
- **Specialized High school** (age from 15 to 18 years): 92 respondents

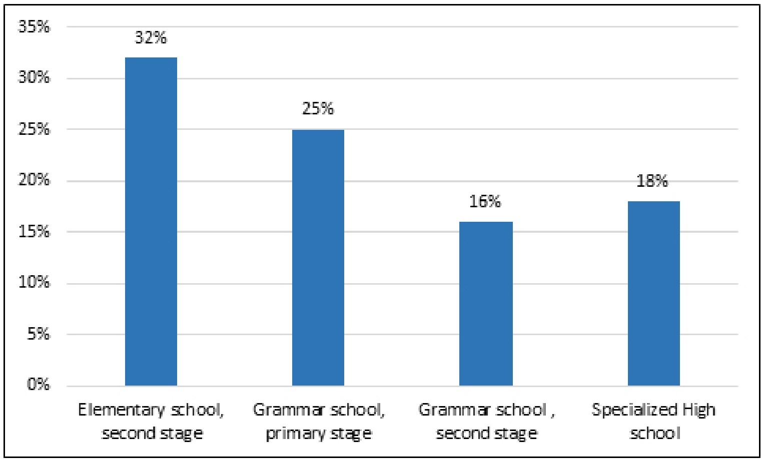

Figure 1. Number of respondents according to types of schools

2.1.2. *Number of respondents: Comparison of Elementary school, second stage and Grammar school, primary stage (see Figure 2).*

Eight-year grammar school:

- prima = 6th grade elementary
- sekunda = 7th grade elementary
- tercie = 8th grade elementary
- kvarta = 9th grade elementary

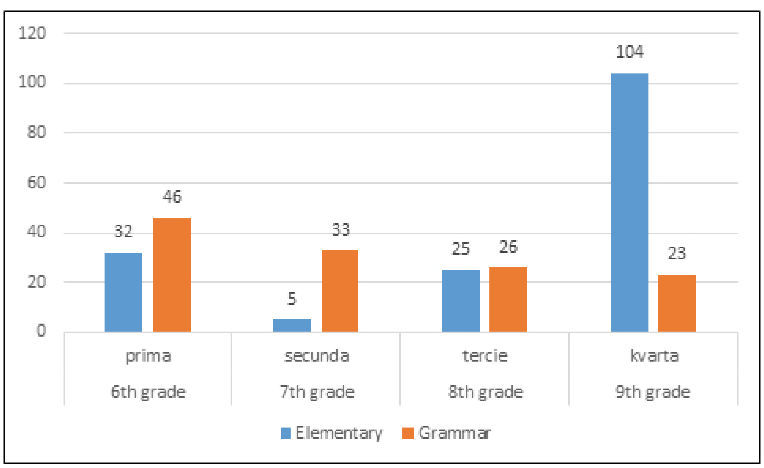

Figure 2. Number of respondents – Comparison of corresponding grades of Elementary and Grammar schools

2.1.3. *Number of respondents: Comparison of Grammar school, second stage and Specialized High school (see Figure 3).*

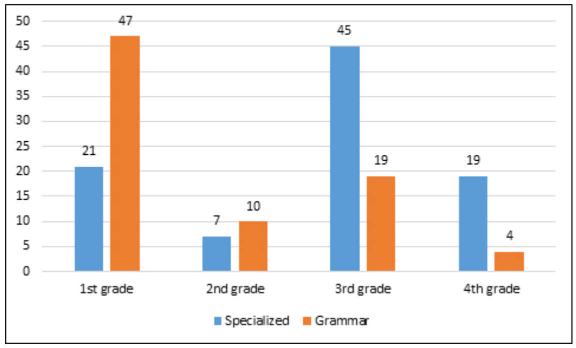

Figure 3. Number of respondents – Comparison of corresponding grades of Grammar school and Specialized High schools.

2.2. **The presentation of the tasks.**

The whole survey consists of 9 tasks. Task processing took place for sample of 466 respondents.

2.2.1. *Write the names of letters of the Greek alphabet.*
The task was to write the correct name of Greek letter according to the Figure 4. in questionnaire

E, ε, ϵ		Θ, θ, ϑ		Ξ, ξ	
Z, ζ		K, κ, $\varkappa$		Σ, σ, ς	
H, η		N, ν		Ψ, ψ	

Figure 4.

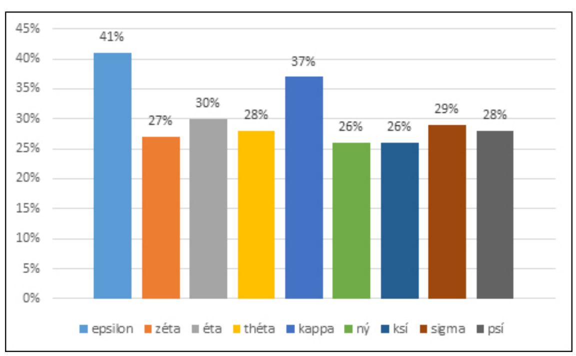

Figure 5. Graphical presentation of the success rate for the names of selected letters

2.2.2. *How to write letters of the Greek alphabet.*
The task was to write the correct symbol of Greek letter according to the Figure 6. in questionnaire

alfa		lambda		tau	
beta		mý		fí	
gama		pí		psí	
delta		ró		omega	

Figure 6.

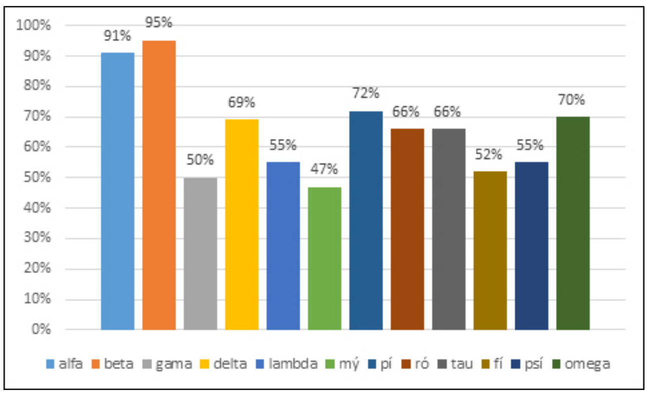

Figure 7. Graphical presentation of the success rate for selected letter symbols

2.2.3. *Names of angles.*
The task was to write the correct name of angels according to Figure 8. (a.–f.) in the questionnaire.

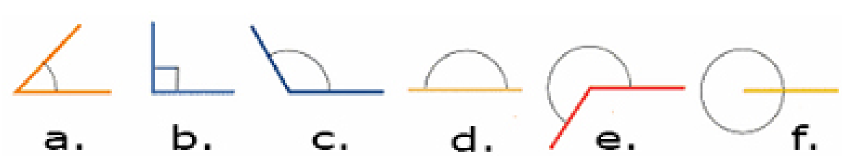

Figure 8.

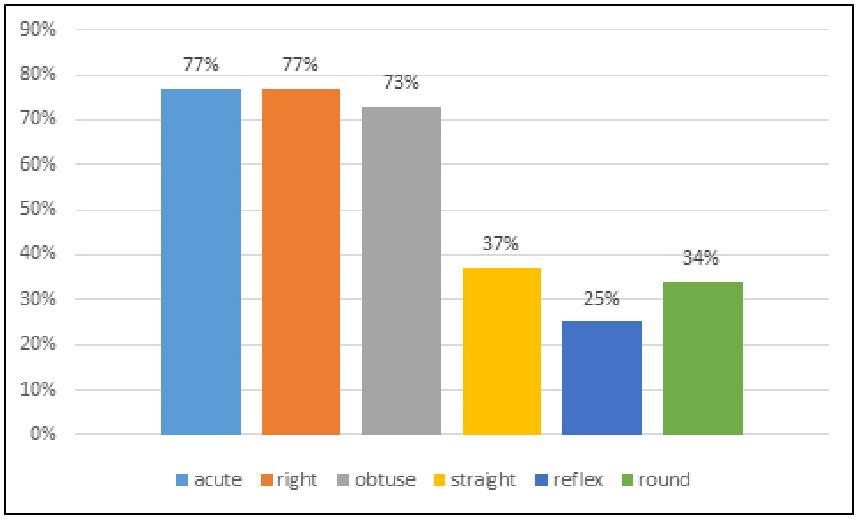

Figure 9. Graphical presentation of the success rate of angle's designation

2.2.4. *Which angle is convex / positively oriented / interior?*
The task was to correctly determine convex angle according to Figure 10. (choice A./B.), positively oriented angle according to Figure 11. (a choice of a/b) and interior angle according to Figure 12. (a choice of a/b) in the questionnaire.

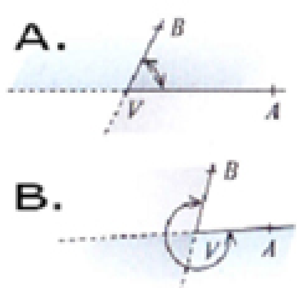

Figure 10.

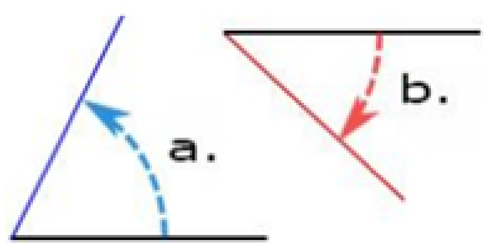

Figure 11.

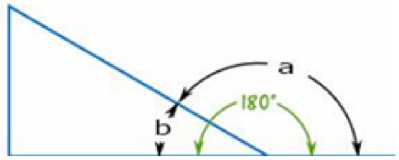

Figure 12.

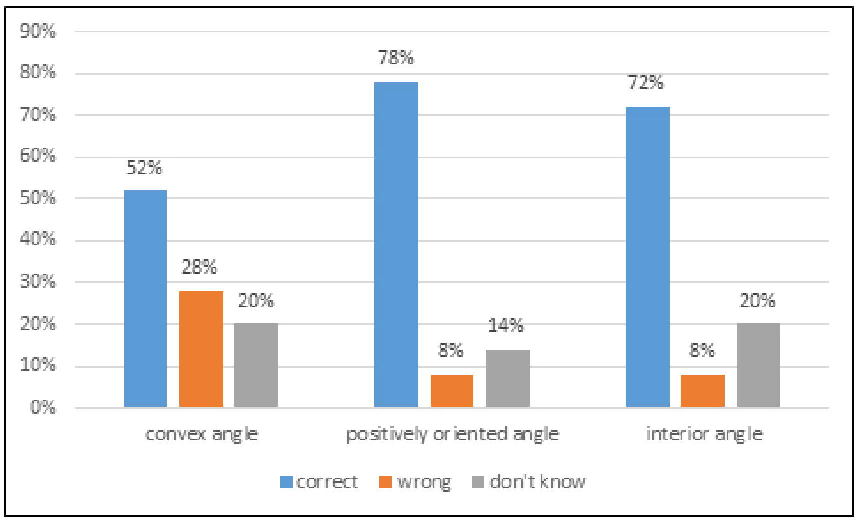

Figure 13. Graphical presentation of the success rate of the correct determination of convex angle, positive oriented angle and interior angle

2.2.5. *Names of Polygons.*
The task was to correctly designate six selected polygons according to Figure 14. in the questionnaire.

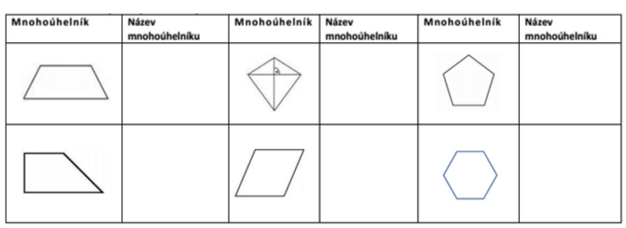

Figure 14.

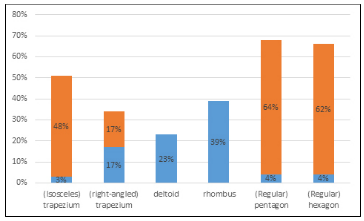

Figure 15. Graphical presentation of the success rate polygon's designation (brackets refer to the blue part of the column)

2.2.6. *Names of 3D objects.*
The task was to correctly designate ten 3D objects according to Figure 16. in the questionnaire.

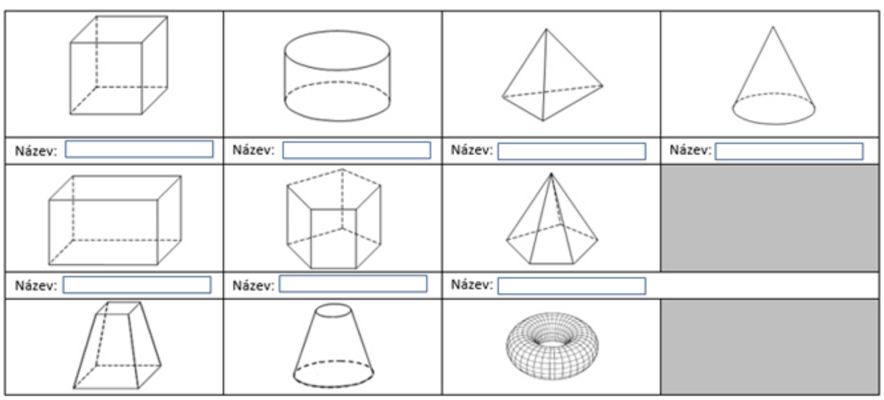

Figure 16.

2.2.7. *Circle and line.*
The task was to correctly designate red signed lines according to Figure 18. in the questionnaire.

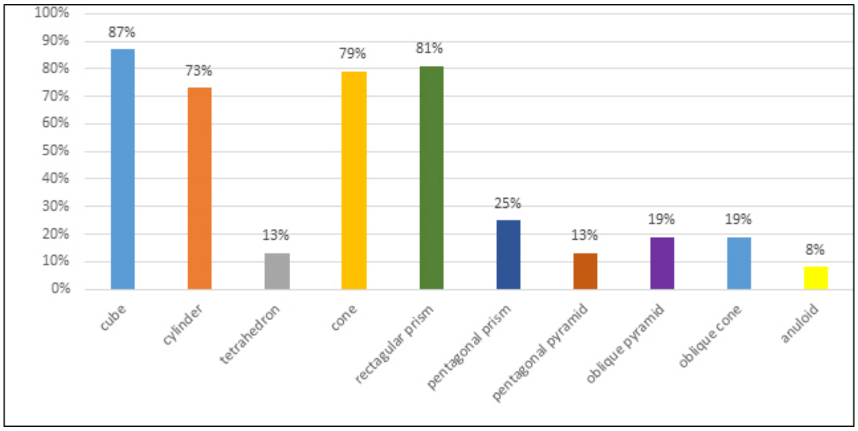

Figure 17. Graphical presentation of the success rate 3D objects designation

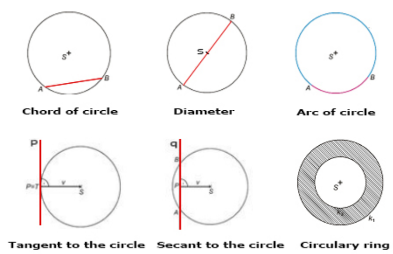

Figure 18.

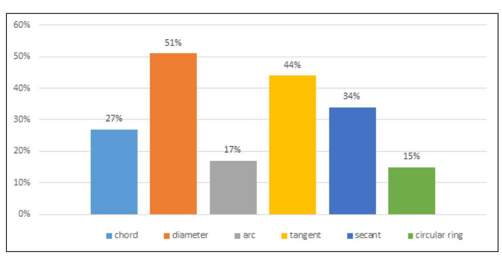

Figure 19. Graphical presentations of the success rate red lines and cross-hatched part of the ring designation

2.2.8. *Visibility of the sides of a cube.*
The task was to correctly determine visibility of the cube's sides in 4 different positions according to Figure 20. – 23. in the questionnaire.

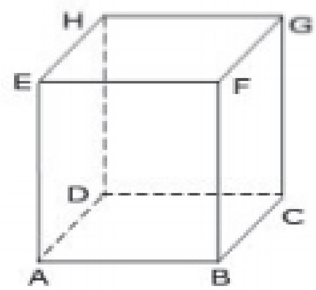

Figure 20. Cube A

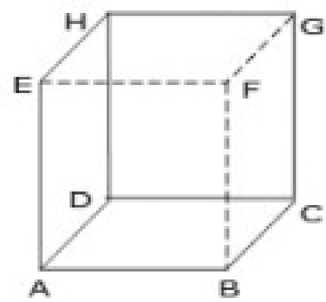

Figure 21. Cube B

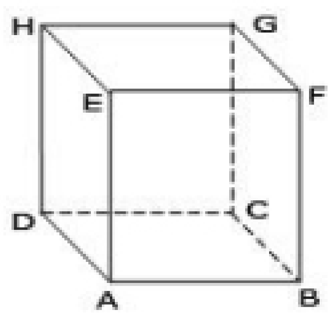

Figure 22. Cube C

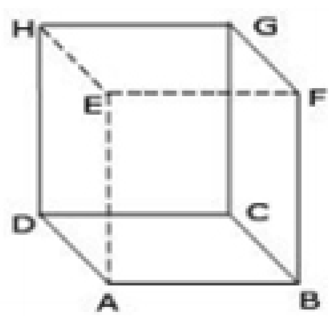

Figure 23. Cube D

Visibility of cube's sides from previous figures is shown as follows:

- Cube A: visibility of sides: ABEF, EFGH, BCFG
- Cube B: visibility of sides: CDGH, ABCD, ADEH
- Cube C: visibility of sides: ABEF, EFGH, ADEH
- Cube D: visibility of sides: CDGH, ABCD, BCFG

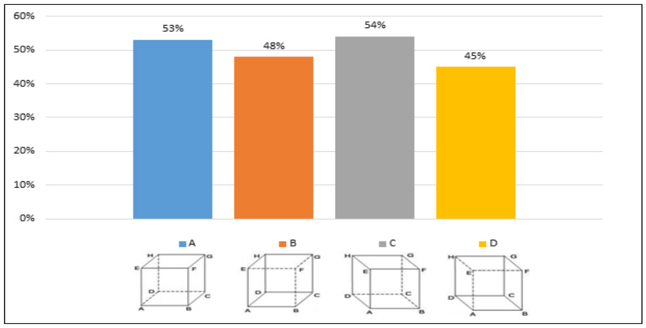

Figure 24. Graphical presentation of the success rate of the correct visibility determination of the cube's sides for its different positions

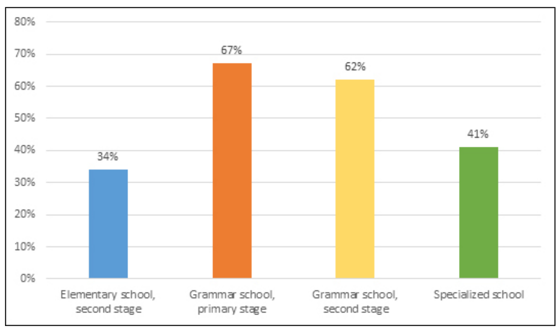

Figure 25. Graphical presentation of the success rate of the correct visibility determination (see Fig. 24) according to types of school

Note: Numerical values in percent mean average values over all space positions of a cube (A – D).

Overview of the presentation of the success rate according to various criteria:

Percent values according to gender (without distinction school types):
Female 50%
Male 45%

Percent values according to gender at Grammar school, 2nd degree:
Female 59%
Male 65%

Percent values according to gender at Specialized High School:
Female 43%

Male 40%

Percent values at Specialized High School for technical school and non-technical school:
Technical school 41%
Non-technical school 42%

2.2.9. *Recognition of parallelism.* Task was to decide, whether it would be possible to consider curves in Fig. 26 (A. – C.) as parallel.

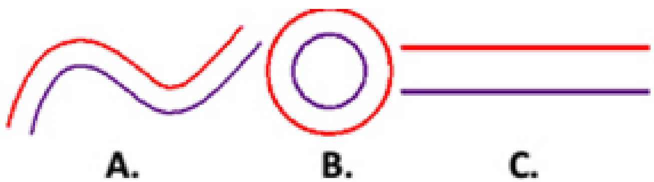

Figure 26.

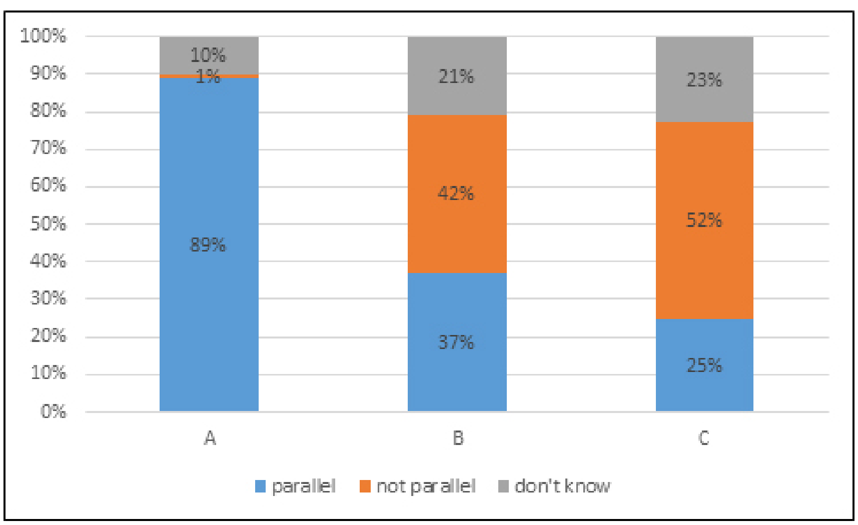

Figure 27. Graphical presentation of the responses to the question about curve's parallelism

3. Final remarks

From the survey results shown in this article the general conclusion can be drawn: the geometrical terminology knowledge of respondents is rather on average rate. Due to importance of accurate and clear introducing of terms, this result is not satisfactory. The construction of geometry demands strong knowledge basis.

The success rate differs not only in particular tasks, but also in particular groups of respondents. It depends on many factors: current grade of education, individual ability of spatial perception, and the period/length of studied subject matters.

Reference

[1] Grebeň R., Molnár J., *Geometrická představivost a terminologie, In: Molnár, J. a kol.: Geometrická představivost*, Vyd. UP, Olomouc, 2014.

[2] Grebeň R., Molnár J., *Geometrické představy žáků střední odborné školy, In: Profesní příprava učitelů přírodovědných oborů*, Vyd. UP, Olomouc, 2013.

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Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XIX (2014)
51–55

CENTRAL LIMIT THEOREM VISUALIZED IN EXCEL

JITKA HODAŇOVÁ, TOMÁŠ ZDRÁHAL

Abstract

The Central Limit Theorem states that regardless of the underlying distribution, the distribution of the sample means approaches normality as the sample size increases. This paper describes the steps in MS Excel to help students' better understanding of this theorem.

1. Introduction

The Central Limit Theorem (CLT) plays without any doubt a fundamental role in statistical inference. According our experiences neither a formal proof of this theorem nor the active understanding CLT included its application are for most undergraduate students understandable. Even (we will not speak in this paper about the proof of CLT at all) if there are many means that can help to better understand this theorem nowadays – we are speaking about both the computer dynamic software with graphical and simulation capabilities and Internet applet. One of the main reasons is the fact that the general concept of a theoretical sampling distribution seems to be too abstract for students. The most important aspect of CLT is that no conditions are required on the population from which one is sampling. And moreover, the population is (theoretically) infinite. However, students need, at the beginning, to draw a random sample from a finite population with a known distribution and then to compare the sample mean. They have to understand the difference between the population mean and the mean of the sample means – they need real experience with sampling. Further, the Central LIMIT Theorem – just a limit concept should play the first negative role in the understanding of CLT. In this concept there are no restrictions

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on a finite computation that gives a definite answer – this is a difference between an action and a process. And students are used to actions first.

2. Main results

The aim of this paper is to visualize CLT on the finite population.

- We will demonstrate CLT by means of Microsoft Excel – so every student could make similar demonstration himself without any programming knowledge. To enable students benefit from this paper we start with some notions.
- Population parameter: is a numerical descriptive measure of a population.
- Sample statistic is a numerical descriptive measure calculated from sample data.
- Sampling distribution of a sample statistic calculated from a sample of n measurements is the probability distribution of values taken by the statistic in all possible samples of size n taken from the same population. To estimate the unknown value of μ, the sample mean $\overline{x}$ is often used. A sampling distribution can be thought of as a relative frequency distribution with a very large number of samples.

The CLT states that:
Given a population with a finite mean μ and a finite non-zero variance σ^2. Then the sampling distribution of the mean approaches a normal distribution with a mean of μ and a variance of σ^2/N as N, the sample size, increases.

- Confidence interval is the interval which has a pre-specified probability of containing the parameter. To obtain this confidence interval we need to know the sampling distribution of the estimate. Once we know the distribution, we can talk about confidence.

The idea behind confidence intervals is that it is not enough just using sample mean to estimate the population mean. The sample mean by itself is a single point. This does not give any idea as to how good our estimation is of the population mean.If we want to assess the accuracy of this estimate we will use confidence intervals. The all theory behind the CLT and confidence intervals was created as follows. The finite set was taken and all samples were generated – for the finite set of the cardinality N and samples of the cardinality k there are $\binom{N}{k}$ (combinations) samples together (sampling without replacement). It is nothing than easy computation to find that the population mean equals to the mean of sampling means. And this is the idea of the point estimation and subsequently the confidence intervals.

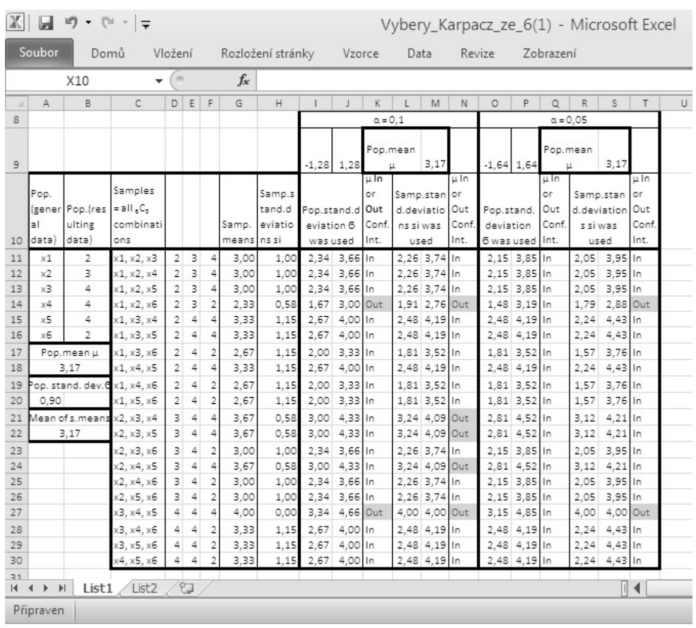
All this notions and claims are illustrated by means of the charts in Microsoft Excel. Because the only aim is students to better understand the CLT we use the methods concerning the Normal Distribution regardless we should use t distribution or even non-parametric methods – because our sample size is very small. (We investigate population size 6 by means of all samples size 3. The larger numbers would make impossible to demonstrate our investigation clearly in this printed form.)

There are two figures showing two different situations. Because the excel sheet is interactive in the way that resulting data are randomly taken from the set $\{1, 2, 3, 4\}$, we can immediately pressing F9 key get another resulting data.

3. Final remarks

After studying figures bellow students can understand how the fact "population mean is In or Out of the confidence interval" depends on:

(1) Where standard deviation comes from (population standard deviation – we do not know it in practice or sample standard deviation – we can compute it always).
(2) Alpha level (common is $\alpha = 0,5$, we use also $\alpha = 0,1$ only because the understanding of its role by using confidence intervals).



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
8									α = 0,1						α = 0,05					
9									-1,28	1,28	Pop.mean μ		3,17		-1,64	1,64	Pop.mean μ		3,17	
10	Pop. (general data)	Pop.(resulting data)	Samples = all ₆C₃ combinations				Samp. means	Samp.stand.deviations si	Pop.stand.deviation σ was used		μ In or Out Conf. Int.	Samp.stand.deviations si was used		μ In or Out Conf. Int.	Pop.stand. deviation σ was used		μ In or Out Conf. Int.	Samp.stand.deviations si was used		μ In or Out Conf. Int.
11	x1	2	x1, x2, x3	2	3	4	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
12	x2	3	x1, x2, x4	2	3	4	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
13	x3	4	x1, x2, x5	2	3	4	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
14	x4	4	x1, x2, x6	2	3	2	2,33	0,58	1,67	3,00	Out	1,91	2,76	Out	1,48	3,19	In	1,79	2,88	Out
15	x5	4	x1, x3, x4	2	4	4	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In
16	x6	2	x1, x3, x5	2	4	4	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In
17	Pop.mean μ		x1, x3, x6	2	4	2	2,67	1,15	2,00	3,33	In	1,81	3,52	In	1,81	3,52	In	1,57	3,76	In
18	3,17		x1, x4, x5	2	4	4	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In
19	Pop. stand. dev.σ		x1, x4, x6	2	4	2	2,67	1,15	2,00	3,33	In	1,81	3,52	In	1,81	3,52	In	1,57	3,76	In
20	0,90		x1, x5, x6	2	4	2	2,67	1,15	2,00	3,33	In	1,81	3,52	In	1,81	3,52	In	1,57	3,76	In
21	Mean of s.means		x2, x3, x4	3	4	4	3,67	0,58	3,00	4,33	In	3,24	4,09	Out	2,81	4,52	In	3,12	4,21	In
22	3,17		x2, x3, x5	3	4	4	3,67	0,58	3,00	4,33	In	3,24	4,09	Out	2,81	4,52	In	3,12	4,21	In
23			x2, x3, x6	3	4	2	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
24			x2, x4, x5	3	4	4	3,67	0,58	3,00	4,33	In	3,24	4,09	Out	2,81	4,52	In	3,12	4,21	In
25			x2, x4, x6	3	4	2	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
26			x2, x5, x6	3	4	2	3,00	1,00	2,34	3,66	In	2,26	3,74	In	2,15	3,85	In	2,05	3,95	In
27			x3, x4, x5	4	4	4	4,00	0,00	3,34	4,66	Out	4,00	4,00	Out	3,15	4,85	In	4,00	4,00	Out
28			x3, x4, x6	4	4	2	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In
29			x3, x5, x6	4	4	2	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In
30			x4, x5, x6	4	4	2	3,33	1,15	2,67	4,00	In	2,48	4,19	In	2,48	4,19	In	2,24	4,43	In

Figure 1

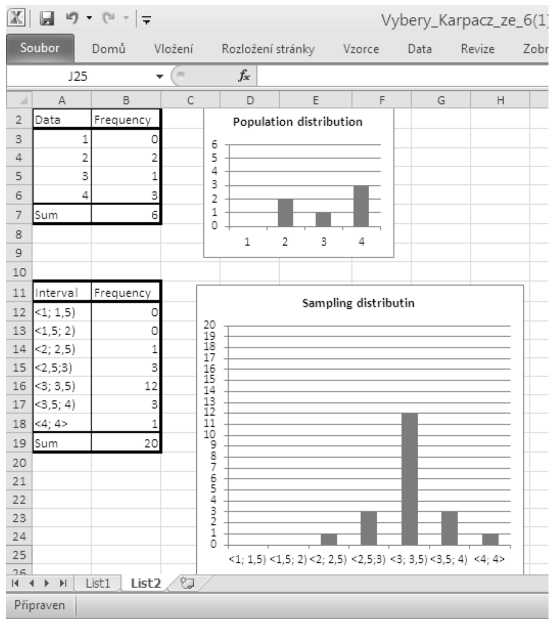
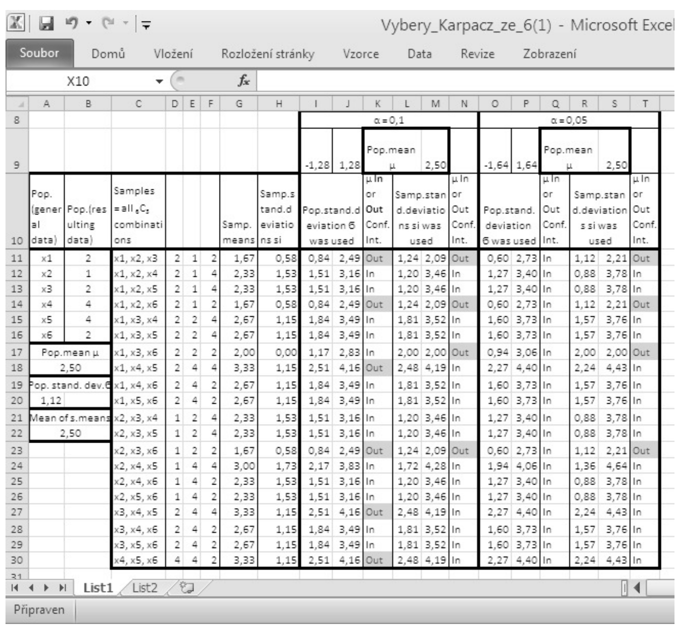


FIGURE 2


Vybery_Karpacz_ze_6(1) - Microsoft Excel

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8									α = 0,1						α = 0,05					
9									-1,28	1,28	Pop.mean μ		2,50		-1,64	1,64	Pop.mean μ		2,50	
10	Pop. (general data)	Pop.(resulting data)	Samples = all ${}_6C_3$ combinations				Samp. means	Samp.stand.deviations si	Pop.stand.deviation σ was used		μ In or Out Conf. Int.	Samp.stand.deviations si was used		μ In or Out Conf. Int.	Pop.stand. deviation σ was used		μ In or Out Conf. Int.	Samp.stand.deviations si was used		μ In or Out Conf. Int.
11	x1	2	x1, x2, x3	2	1	2	1,67	0,58	0,84	2,49	Out	1,24	2,09	Out	0,60	2,73	In	1,12	2,21	Out
12	x2	1	x1, x2, x4	2	1	4	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
13	x3	2	x1, x2, x5	2	1	4	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
14	x4	4	x1, x2, x6	2	1	2	1,67	0,58	0,84	2,49	Out	1,24	2,09	Out	0,60	2,73	In	1,12	2,21	Out
15	x5	4	x1, x3, x4	2	2	4	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
16	x6	2	x1, x3, x5	2	2	4	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
17	Pop.mean μ		x1, x3, x6	2	2	2	2,00	0,00	1,17	2,83	In	2,00	2,00	Out	0,94	3,06	In	2,00	2,00	Out
18	2,50		x1, x4, x5	2	4	4	3,33	1,15	2,51	4,16	Out	2,48	4,19	In	2,27	4,40	In	2,24	4,43	In
19	Pop. stand. dev.σ		x1, x4, x6	2	4	2	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
20	1,12		x1, x5, x6	2	4	2	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
21	Mean of s.means		x2, x3, x4	1	2	4	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
22	2,50		x2, x3, x5	1	2	4	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
23			x2, x3, x6	1	2	2	1,67	0,58	0,84	2,49	Out	1,24	2,09	Out	0,60	2,73	In	1,12	2,21	Out
24			x2, x4, x5	1	4	4	3,00	1,73	2,17	3,83	In	1,72	4,28	In	1,94	4,06	In	1,36	4,64	In
25			x2, x4, x6	1	4	2	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
26			x2, x5, x6	1	4	2	2,33	1,53	1,51	3,16	In	1,20	3,46	In	1,27	3,40	In	0,88	3,78	In
27			x3, x4, x5	2	4	4	3,33	1,15	2,51	4,16	Out	2,48	4,19	In	2,27	4,40	In	2,24	4,43	In
28			x3, x4, x6	2	4	2	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
29			x3, x5, x6	2	4	2	2,67	1,15	1,84	3,49	In	1,81	3,52	In	1,60	3,73	In	1,57	3,76	In
30			x4, x5, x6	4	4	2	3,33	1,15	2,51	4,16	Out	2,48	4,19	In	2,27	4,40	In	2,24	4,43	In

FIGURE 3

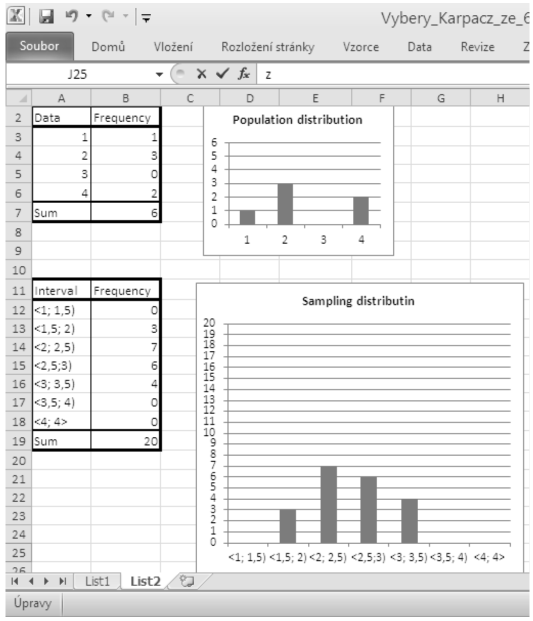

FIGURE 4

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ICT INFLUENCE ON PUPILS' LEARNING (NOT ONLY IN MATHEMATICS TEACHING)

LENKA JANSKÁ

Abstract

The article is concerned with the influence of modern technologies on learning mathematics and other subjects. It presents Marc Prensky's theory of digital natives and digital immigrants in the roles of the present-day pupils and teachers. The article includes opinions with regard to the necessary change in the educational process, especially in the approach of teachers to the present-day pupils and students.

1. Introduction

The human society has always been under constant development. This development is influenced not only by the change of the environment, social aspects and scientific knowledge, but also by the continually developing technology around us, as a matter of course. After the arrival of the first computers, no one expected that computers would one day be so small and (relatively) cheap that nearly everybody would be able to purchase one. A few years later, as home computers started to develop and be used by the general public, hardly anyone could imagine that we would all have mobile phones by means of which we would always be able to stay in touch nearly anywhere in the world. All these technological inventions have gradually become natural parts of most of us, and we cannot imagine our lives without computers, mobile phones or the Internet. However, this relatively rapid change has also brought about a change in many other areas. The abovementioned technologies are primarily supposed to help us and facilitate both work and entertainment. It is thus obvious that such technological innovations have altered various aspects of our lives, including education.

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2. Main results

According to Prensky (Prensky, 2001), the present-day students have not only changed the way in which they behave, dress and communicate; they have also altered the way in which they think and perceive things, which is strongly dependent on modern technologies that surround them ever since they are born. Today's high school graduates spend twice as much time playing computer games and four times as much time watching television as they spend reading. This is a significant change which Prensky calls discontinuity, thus expressing the difference between the present-day students and previous generations. The new approach to this generation of students is based on the fact that they are different. Some authors call them the N-generation (Oblinger, Oblinger), others call them the D-generation (Palfrey, Glaser); however, according to Prensky, the most accurate term is "digital natives", that is to say those who were born in the digital world of computers, video games and the Internet. Prensky calls the other group "digital immigrants".

Digital natives are people who grow up in an environment full of modern technologies, such as computers, digital music players, video cameras, web cameras and mobile phones. The main difference between generations lies in the way people think and process information. Digital natives are used to receiving information very quickly and prefer parallel activities and multitasking; they also prefer graphic depiction over text and play over "serious" work. They like network cooperation and chance access to information (hypertext). They expect immediate praise and frequent appreciation of their work. They do not see computers, mobile phones or the Internet as modern digital technologies; to them, such technologies are an integral part of their lives.

Digital immigrants are members of older generations, who learned to use the above technological devices only later in life. To them, therefore, technologies are something new, unnatural and even unnecessary. Immigrants try to adapt to the new environment but they always retain a part of their original environment (accent). The digital accent of digital immigrants is manifested in various ways, for example: they use the Internet as a secondary source of information (they still prefer printed materials), they study manuals and instructions first instead of just operating devices intuitively, they print out e-mail messages and documents, they make phone calls to ask whether the person has received their e-mail message etc. Digital immigrants do not use the possibilities and ways of work like natives. They do not believe that digital natives can learn properly while watching television or listening to music because digital immigrants never did so themselves.

Teachers wrongly assume that today's students are not any different from themselves at the time when they were students, and can therefore use the same well-established methods by means of which the teachers themselves used to learn. Flexible digital immigrants will understand that their students will always be better in this respect, and will use this situation to introduce a higher-quality educational process. Others will be dissatisfied with the participants of the educational process and with the process itself, and will just criticize the new ways and think back to their own student years. The approach will never change, though. Therefore, if we want to educate digital natives in an adequate manner, the issue must be addressed. With respect to the methods and content of learning, this means that today's teachers should be able to communicate in the language and style used by today's students without changing the content of lessons or the time-proven way of thinking. Teachers should learn to proceed faster, multi-task, apply randomly obtained information and gradually break away from always working logically small step by small step (Prensky, 2001).

As regards the contextual aspect of competencies, there are two types – the past content and the future content. The past content, required in the past and inherited from the past, includes reading, writing, counting, logical reasoning, and understanding the content of a written text and ideas relating to the past era, that is to say everything that is part of the traditional curriculum. Such content is still important but is associated with a different time period. Some of it will always be crucial (such as logical reasoning); nevertheless, the importance of other areas (such as Euclidean geometry) will decrease, just as it was the case with Latin and Greek. The future content, designated for the present-day and future students, is digital and technological. It includes not only software, hardware, robotics, nanotechnology etc., but also ethics, politics, sociology, foreign languages and other related areas. The future content is highly interesting to students. However, the question is how many teachers, digital immigrants, are able to use it or apply it in teaching practice (Prensky, 2001).

"Every teacher should take into account the environment in which their students live. Teachers in general should realize that students have changed in a certain way, that they live in a perfectly interconnected network with lots of outside contacts, and that it is difficult for students to understand why they should learn by heart something which can be found within seconds. Therefore, it is necessary to connect teaching with practical issues which interest pupils and students. This approach should be prevalent in all components of education and the entire school environment. Only then will students have their own initiative and understand the need for life-long learning, i.e. the 21st-century competency" (Brdička, 2009).

As teachers, we must think of new ways to teach the past and future contents simultaneously and at the same time use the language of the digital natives. In practice, this means we should considerably change the methodology, content and way of thinking. It is hard to say whether it is more difficult to teach new content or to apply new methods to teaching old content. It is thus necessary to find ways to achieve this objective (Prensky, 2001).

3. Final remarks

It goes without saying that the difference between these groups of people – digital natives and digital immigrants – is manifested in many aspects of life and affects a number of factors. It is therefore essential to address this issue and incorporate these changes and development into the educational process so that it remains up-to-date and takes into account the current trends and needs of the society.

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A MISTAKE IN DEFINITION OF A LIMIT OF A FUNCTION AND SOME CONSEQUENCES

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Abstract

Many times our students make some errors in definitions, especially when we must apply some quantifiers. The definition of a limit is one of the definition with many quantifiers, so one can observe many mistakes in it. We want to present one of possible mistakes and show how to improve the understanding of this difficult but one of most important notions.

1. Some basic notions

We shall consider only real functions defined in an open interval.

Let us start from the classical Heine's definition of limit of a real function of a real variable.

Definition 1. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0, if for each converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0, the sequence $\left(f\left(x_n\right)\right)_{n=1}^{\infty}$ is convergent to g.*

In other words one can formulate this definition as follows:

Definition 2. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0 if for each sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) if*

(1) $\forall_{n\in\mathbb{N}}\,(x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

then $\lim\limits_{n\to\infty} f(x_n) = g$.

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2. Mistake

Many times we can hear (from our students) this definition formulated in the following form:

Definition 3. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is a limit of f at point x_0 if there exists a converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0 the sequence $(f(x_n))_{n=1}^{\infty}$ is convergent to the number g.*

And symbolically:

Definition 4. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0 if there exists a sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) such that*

(1) $\forall_{n\in\mathbb{N}}\,(x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

and $\lim\limits_{n\to\infty} f(x_n) = g$.

Of course, this definition is not proper for mathematical analysis. But what can we do if after all someone formulates the definition in that form. If it happens, we can continue our lecture, for example, in this way.

3. Consequences

Let us see what happens if we take this definition into considerations. Define function f like this:

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

Then numbers 0 and 1 are limits of this function at 0!(???) How is it possible. Everybody heard that there is only one limit of a function at a point!

It can happen even worse: Let ϕ be defined in the following way:

$$\phi(x) = \begin{cases} \sin\frac{\pi}{x}, & \text{if} \quad x \in \mathbb{R}\setminus\{0\}, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Now, any point from the interval $[-1,1]$ is a limit number of function ϕ.

The answer is rather strange, we are accustomed to the fact that any function has only a unique limit if it has a limit.

So we are mistaken. But never mind let's continue our lesson. Suppose, we want to know what happens if such a notion has its own name. Then let us call this notion as limit number (or limit point). In that case we take:

Definition 5. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called somehow, say it is a limit number of f at point x_0 if there exists a sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) such that*

(1) $\forall_{n\in\mathbb{N}}\,(x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

and

(3) $\lim\limits_{n\to\infty} f(x_n) = g$.

Let $f\colon (a,b) \longrightarrow \mathbb{R}$ be any function and x_0 be a point from (a,b). Consider any converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0.

There are two possibilities:
1). The sequence $\left(f\left(x_n\right)\right)_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\left(f\left(x_{k_n}\right)\right)_{n=1}^{\infty}$ which is convergent to some real number, say g. Of course, the subsequence $\left(x_{k_n}\right)_{n=1}\infty$ of the sequence $(x_n)_{n=1}^{\infty}$ is also convergent to x_0, hence g is a limit number of function f at the point x_0.
2). The sequence $\left(f\left(x_n\right)\right)_{n=1}^{\infty}$ is unbounded, for example it is unbounded from above. Then $+\infty$ is a limit number of f at the point x_0.

In this way we proved that each function has a limit number at any point of (a,b).

4. What next?

Considering Heine's method of defining limits, one can ask whether it is possible to give adequate Cauchy's condition for this notion. Without great difficulty one can get the following characterization for the case when g is a real number and x_0 is real number as well.

Theorem 1. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then a real number g is a limit number of f at point x_0 if and only if*

$$\forall_{\epsilon>0}\forall_{\delta>0}\exists_{x\in(a,b)}\Big(\big(x\in(x_0-\delta,x_0+\delta)\setminus\{x_0\}\big)\wedge f(x)\in(g-\epsilon,g+\epsilon)\Big)$$

While proving that equivalence it is time and place to remind Axiom of Choice and its importance in modern mathematics.[1]

It is important in that case, that this axiom is needed in the proof into one direction. The implication that Cauchy's condition implies Heine's condition does need assumption of Axiom of Choice. But the inverse implication makes use of that axiom.

[1]Probably, Wacław Sierpiński was the first mathematician, who had found out that the Axiom of Choice is necessary to prove the equivalence of Heine's and Cauchy's conditions of limit of a function.

5. What would happen further?

Our next considerations would follow in this way:

- Is it possible to characterize limit numbers in Cauchy's manner in the case when x_0 equals plus or minus infinity?
- Is it possible to characterize limit numbers in the case when g equals plus or minus infinity? If so:
- Characterize all other limit numbers by Cauchy's manner.
- Prove that the set of all limit numbers of any function f at any point x_0 is closed.
- Prove that a function has a limit at a point if and only if the set of all limit numbers at that point is a singleton.
- Define left sided and right sided limit numbers.
- Prove Young's theorem of asymmetry, i.e. that the set of points x at which the set of left sided limit numbers is different from the set of right-sided limit numbers is at most countable.
- Define upper and lower limits, applying the notion of limit numbers.
- Define Baire's upper and lower function of a function f.
- Prove that the set of all points of continuity of any function is G_δ-set.

Summing up, coming from an erroneous definition we were able to expand the theory of real functions and improve the understanding of a notion of a limit. So the mistake and further considerations did not waste our time. Moreover, such considerations can improve understanding of the notion of a limit.

Some of the generalizations of this notion can be found in the following articles:

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WHY THE AREA OF A RECTANGLE IS CALCULATED BY THE FORMULA $P = a \cdot b$?

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Abstract

It is evident for many people that the area of a rectangle can be calculated according to the very well known formula:

$$P = a \cdot b.$$

We, mathematicians do believe in no statement without the proof. Then we can ask whether it is possible to prove that this formula is correct. This article answers to that question.

1. Introduction

First of all we should discuss the problem which conditions are necessary for a function to represent area of geometric figures. Such functions are called „measures". We suppose that everybody knows what the length of a segment (of the straight line) is. We shall return to that problem soon. However, we have to define conditions for „measure" first.

With no doubt, we can say that the measure of any area should be a positive real number. Moreover, if a rectangle is divided onto two disjoint rectangles, then the area of it should be the sum of its smaller parts. Thus we can define a measure (in some kind of class of subsets of a specific space). For further information on measure theory see [1].

Definition 1. *Let $\mathcal{S}$ be a class of subsets of a fixed set X. The class $\mathcal{S}$ is called a field of sets if it fulfils the following conditions:*

(1) $\varnothing \in \mathcal{S}$,
(2) *if* $A \in \mathcal{S}$ *and* $B \in \mathcal{S}$, *then* $A \cup B \in \mathcal{S}$,
(3) *if* $A \in \mathcal{S}$, *then* $X \setminus A \in \mathcal{S}$.

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Definition 2. *Let $\mathcal{S}$ be a field of subsets of a fixed set X. A function $\mu\colon \mathcal{S} \longrightarrow \mathbb{R}$ is called a measure in $\mathcal{S}$ if it fulfils the following conditions:*

(4) $\mu(\varnothing) = 0$,

(5) *if $A \in \mathcal{S}$, $B \in \mathcal{S}$ and $A \cap B = \varnothing$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.*

2. A few properties of fields of sets and measure in $\mathbb{R}$

If $X = \mathbb{R}$ (or equivalently X is a straight line), then we also require that a measure should be invariant with respect to translation. For example, the intervals $[0, 1]$ and $[4, 5]$ should have the same measure. One can prove that there is no measure invariant under translation for which $\mathcal{S}$ equals to the class of all subsets of $\mathbb{R}$. Of course, there are some fields of subsets of the set of real numbers which fulfil our requirements. Even more, there exists the smallest field of subsets of $\mathbb{R}$ which contains all intervals. For such fields there exist measures fulfilling our requirements.

The same remarks are true if we replace $\mathbb{R}$ by $\mathbb{R}^2$ and intervals by rectangles.

If $A \subset \mathbb{R}$ and $q \in \mathbb{R}$, then by $A + q$ we denote the set

$$\big\{x \in \mathbb{R}\colon \exists_{a\in A}\,(x = a + q)\big\}$$

Definition 3. *If a measure μ in $\mathbb{R}$ or in $\mathbb{R}^2$ fulfils the condition*

(6) *$\mu(A + q) = \mu(A)$ for each set A from $\mathcal{S}$ and each real number q,*

(7) *$\mu([a, b]) = b - a$ for each interval $[a, b]$,*

then μ is called Jordan measure.

Theorem 1. *If $\mathcal{S}$ is a field of sets, A and B belong to $\mathcal{S}$, then $A \setminus B \in \mathcal{S}$.*

Theorem 2. *If μ is a measure (in the sense of Definition 2), $A \in \mathcal{S}$ and $B \in \mathcal{S}$, then*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B).$$

3. Area of rectangles

Coming back to the main problem, let us remark, that the measure of a rectangle is a function which depends on the length of its sides. So, if we consider rectangles which sides are parallel to x and y axes, then the measure of it is a function of two variables. If we denote it by P, then this function has to fulfil the following conditions:

(8) $P(x, y) > 0$ for each positive numbers x and y,

(9) $P(x_1 + x_2, y) = P(x_1, y) + P(x_2, y)$ for each positive numbers x_1, x_2 and y,

(10) $P(x, y_1 + y_2) = P(x, y_1) + P(x, y_2)$ for each positive numbers x, y_1 and y_2.

Similar problem has been considered in [2]. We shall present the necessary calculations only.

Suppose that $P(1, 1) = a$. Of course, $a > 0$. Then the function f defined by

$$f(x) = P(x, 1)$$

fulfils the following conditions:

(11) $f(x) > 0$ for any positive number x,
(12) $f(x_1 + x_2) = f(x_1) + f(x_2)$ for every positive numbers x_1 and x_2,
(13) $f(1) = a$,
(14) if $x_1 < x_2$ then $f(x_1) < f(x_2)$.

Any function fulfilling condition (12) is called additive. Let us consider such additive function.

In view of those conditions we infer that

$$f(2) = 2 \cdot a$$

since

$$f(2) = f(1 + 1) = f1) + f(1) = a + a = 2a$$

and, by induction,

$$f(n) = n \cdot a$$

for every positive integer n.

Similarly,

$$f\left(\frac{1}{2}\right) = \frac{1}{2} \cdot a,$$

since

$$f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f(1) = a.$$

As before, applying mathematical induction one can prove that

$$f\left(\frac{1}{n}\right) = \frac{1}{n} \cdot a$$

for every positive integer n and similarly

$$f\left(\frac{k}{n}\right) = \frac{k}{n} \cdot a$$

for every positive integers k and n.

In this way we have proved that

$$f(q) = q \cdot a$$

for each positive rational number q.

In the end, we are going to prove that

$$f(x) = x \cdot a$$

for each positive real number x. To prove this, let us assume that it is not true, which means that there exists a real (not rational) number x such that

$$f(x) \neq x \cdot a.$$

There are two possibilities:

$$f(x) < x \cdot a \quad \text{or} \quad f(x) > x \cdot a.$$

Let us consider the first case. Since $a > 0$, thus $\frac{f(x)}{a} < x$. Then there exists a rational number w such that

$$\frac{f(x)}{a} < w < x,$$

hence $f(x) < aw$, therefore $f(x) < f(w)$. Thus we get the contradiction with (14).

Similar argumentation can be done in the other case.

In such a way we have proved that $f(x) = ax$ for every positive real number x, where a is a positive constant.

Usually we assume that $a = 1$, but this time we have to consider a as a function of a real variable y i.e. the second side of a rectangle.

If we repeat the same arguments for the second variable y, we infer that the area $P(x, y)$ of the rectangle which sides have lengths x and y is given by the formula:

$$P(x, y) = x \cdot y.$$

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DIVISIBILITY AND ITS APPLICATION IN TEACHING MATHEMATICS IN THE THIRD EDUCATIONAL LEVEL

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Abstract

Elements of theory of divisibility are present in many various interesting mathematical tasks, especially in tasks that are addressed to talented pupils taking part in mathematical competitions. Good understanding of it lets solve very interesting and difficult (at first glance) issues. On the other hand, there are a lot of problems with understanding such terms as: multiple, divisor, divisibility, prime number, LCM, GCD etc. The purpose of the article is presenting the base terms of theory in the language understood for pupil of the gymnasium (3-rd educational level, 13-16 years old). In addition, we present some algorithms that are used to solve problems from the theory of divisibility and we discuss the influence the choice of the algorithm on its effectiveness (so we analyse its complexity). Presented algorithms let us create computer programs that solve the problems mechanically. We also enlarge a bit some topics for those ones which can be taught in the class of pupils which are interested in Mathematics.

1. Divisibility and multiples

Everybody meets divisibility in the second educational level. Children are taught that some positive integers can be divided by some positive integers, but some of them not. We can say that a positive integer n can be divided by a positive integer m with a remainder. It means that for n and m there exist integers k and r such that

$$\frac{n}{m} = k + \frac{r}{m}, \quad \text{where} \quad r = 0, 1, \ldots, m-1.$$

In other words:

$$n = k \cdot m + r, \quad \text{where} \quad r = 0, 1, \ldots, m-1.$$

If we define divisibility in the following way: a number m is a divisor of a number n if the remainder from the division n by m equals 0, then the

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word *divisor* has some connection with the word *divide*. In fact, we apply here the following theorem:

Theorem 1. *For any nonnegative integer n and positive integer m there exist nonnegative integers k and r such that*

$$n = k \cdot m + r, \quad \text{where} \quad r = 0, 1, \ldots, m-1.$$

If the remainder equals 0, then this theorem says that

$$n = k \cdot m.$$

And this equality plays the main role in the definition of divisibility on the higher levels. The students are forgetting the prime idea of divisibility, so it can be one of the mistakes in understanding this notion defined in the following way:

Definition 1. *An integer n is divisible by positive integer m if there exists an integer k such that $n = k \cdot m$. In that case m is called a divisor of n.*

This definition can be generalized for all integers, for elements from some rings and so on. It is pretty useful, however can cause some problems for our students since the word "divisible" is similar (especially in polish language) to the word "division". But there is no division, multiplication only, in this definition.

One could state from the previous definition that 0 would be divisible by 0 however there is no result of the operation $0 : 0$! That is why the number m must be positive.

For example 12 is divisible by 4 and by 6; $(12 = 4 \cdot 3,\ 12 = 6 \cdot 2)$, but 12 is not divisible by 5; (there exists no positive integer k such that $12 = 5 \cdot k$).

Definition 2. *For a nonnegative integer n, every product $k \cdot n$, where k is a nonnegative integer, is called the multiple of the number n.*

The set of all multiples of n is denoted as W_n. For example:
$W_0 = \{0\}$
$W_1 = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$
$W_3 = \{0, 3, 6, 9, \ldots\} = 3\mathbb{N}$ – numbers divisible by 3
$W_{11} = \{0, 11, 22, 33, \ldots\} = 11\mathbb{N}$ – numbers divisible by 11

It is very important (and often forgotten in schoolbooks) that 0 is the least multiple of any positive integer n.

Remark 1. *Since we have restricted our considerations to positive integers, thus the considered notions have close connections.*

If a positive integer m is a divisor of the positive integer n, then n is a multiple of m and conversely, if a positive integer n is a multiple of a positive integer m, then m is a divisor of n.

Divisors of any positive integer can be characterized by the idea of congruence modulo m.

Theorem 2. *If n is a nonnegative integer, then every integer m, which satisfies condition $n = 0$ modulo m, is a divisor of the number n.*

The set of all divisors of the number n is denoted as D_n. For example:
$D_0 = \{1, 2, 3, \ldots\} = \mathbb{N}_+$,
$D_1 = \{1\}$,
$D_3 = \{1, 3\}$,
$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.
Of course, 1 is a divisor of every positive integer and n is the divisor of n, so every positive integer greater than 1 has at least two divisors.

2. Prime and composed numbers

Definition 3. *A positive integer is called prime number if it is greater than 1 and it is divisible by 1 and itself only (it has only two divisors).*

Definition 4. *A positive integer n is called composed if it is not prime.*

Let us notice that the above definitions have sense only for positive integers greater than 1. Moreover, composed number has a divisor greater than 1 and smaller than n (it has at least three divisors). Hence a positive integer n is composed if it has divisor greater than 1 and not greater than $\lfloor\sqrt{n}\rfloor$. Then we have to search divisors only to $\lfloor\sqrt{n}\rfloor$.
It is not difficult to prove that:

Theorem 3. *The set of prime numbers in infinite.*

The proof of this theorem is quite simple: If this set would be finite, then it would be represented as a finite sequence $(a_1, \ldots, a_n)$. Then the number $a_1 \cdot \ldots \cdot a_n + 1$ would be prime, since it would not be divisible by any of all previous prime numbers. A contradiction proves the theorem.

Theorem 4. *Every composed positive integer can be represented in only one way (if we ignore the order) as a product of prime numbers.*

This theorem is connected with the problem of representing any positive integer as a product of prime numbers. But before we are able to represent a number in such a form, it will be useful to find all divisors of a number. It will be also needed to find all prime numbers less than a given number. It is possible to write small computer program, which can find prime numbers (less than a given number). We present such programs written using Python language.

3. Sieve of Eratosthenes

The method of Eratosthenes is one of the best and fastest methods of finding prime numbers. It is also important because of it is easy writing computer programs finding prime numbers smaller than n. This algorithm has complexity $O(n \log n)$ so it is linear-logarithmic but algorithms finding prime numbers via checking all divisors of positive integers less than n usually has complexity $O(n\sqrt{n})$. Let us show both of those algorithms and examples of programs writing in PYTHON language. Here you can find schemes and programs for finding prime numbers less than a given number n. In all programs and schemes operation $a\%b$ means reminder from the division a by b. For example, $6\%4 = 2$.

```
FINDING PRIME NUMBERS - SLOW PROGRAM IN PYTHON LANGUAGE
import math
print("Determine n")
n=int(input())
for i in range (2,n):
   d=2
   while i%d!=0 and d<=math.floor(math.sqrt(i)):
      d=d+1
   if d==math.floor(math.sqrt(i))+1:
      print(i)
input()
```

```
SIEVE OF ERATOSTHENES - PROGRAM IN PYTHON LANGUAGE
print("Determine n")
n=int(input())
a=[0 for i in range (n)]
for i in range (2,n):
   if a[i]==0:
      print(i)
      k=2
      while (i*k)<n:
         a[i*k]=1
         k=k+1
input()
```

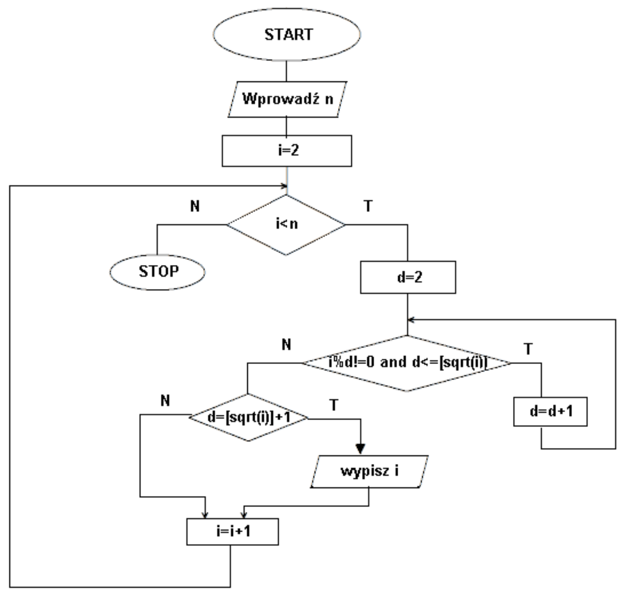

FIGURE 1. **Finding prime numbers – slow algorithm**

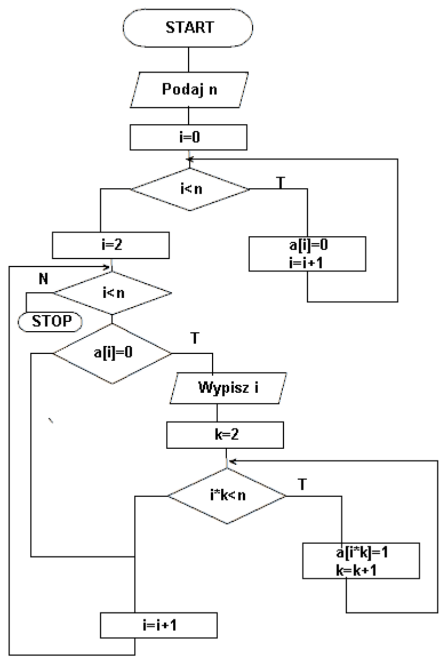

FIGURE 2. **Finding prime numbers – Sieve of Eratosthenes**

4. Least common multiple and greatest common divisor

Talking on prime and composed numbers it is worth saying about least common multiple and greatest common divisor of two given numbers. We noticed that 0 is the least natural number which is common multiple of any natural number, but it is not a good message for the purpose of common multiples. That is why we consider only positive multiples of positive integers.

Definition 5. *The least common multiple (shortly LCM) of two positive integers is the smallest positive integer that is divisible by both of them.*

Hence
$W_{12} = \{0, 12, 24, 36, 48, 60, 72, \ldots\}$
$W_{16} = \{0, 16, 32, 48, 64, 80, 96, \ldots\}$
Then $LCM(12, 16) = 48$.

In school practice LCM is usually computed by factorization onto prime numbers. For example:

	12	**16**
2|	6	8
2|	3	4
2|	–	2
2|	–	1
3|	1	–

Then $LCM(12, 16) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$.

The method which was presented is not the best one. Much better method is based on the following theorem:

Theorem 5. *Let n and m be two positive integers such that $n \leq m$. If k_0 is the least element of the set $\{k \in \mathbb{N} \colon n | (km)\}$, then $LCM(n, m) = k_0 \cdot m$.*

Example 1. Let us consider numbers 24 and 9. We have $24 \cdot 1 = 24$ but 24 is not divisible by 9, $24 \cdot 2 = 48$ but 48 is not divisible by 9. Finally $24 \cdot 3 = 72$ and 72 is divisible by 9 so $LCM(24, 9) = 72$.

Definition 6. *The greater common divisor (GCD) of two positive integers is the greatest number which is the divisor of both of them.*

For example:
$D_{12} = \{1, 2, 3, 4, 6, 12\}$,
$D_{16} = \{1, 2, 4, 8, 16\}$.
Then $GCD(12, 16) = 4$.

We can observe that this time greatest common divisor of two positive integers is a positive number. It is worth saying that any nonnegative integer is a divisor of 0, thus $GCD(0,n) = n$ for any nonnegative integer n.

There is another theorem which allows to compute LCM with the aid of GCD.

Theorem 6. *If n and m are two positive integers then*

$$LCM(n,m) \cdot GCD(n,m) = n \cdot m.$$

The above equality is equivalent to any one of the following ones:

$$LCM(n,m) = \frac{nm}{GCD(n,m)}$$

$$GCD(n,m) = \frac{nm}{LCM(n,m)}$$

In school practice GCD is usually calculated by factorization onto prime numbers. For example:

	12	**16**
2|	6	8
2|	3	4
2|	–	2
2|	–	1
3|	1	–

Then $GCD(12,16) = 2 \cdot 2 = 4$.

This is not the best way of finding GCD of two numbers, especially if those numbers are pretty large. Much better manner is described by Euclid's algorithm. We remind it in the following.

Let m and n be two positive integers such that $n < m$. Let us denote them in the following way:

$$m = n_0, \quad n = n_1.$$

From theorem of dividing positive integers wih the remainder, there are two nonnegative integers k_1 and n_2 such that

$$n_0 = k_1 \cdot n_1 + n_2, \quad \text{where} \quad n_2 \in \{0, 1, \ldots, n_1 - 1\}.$$

Following that way we can state that there exists a sequence $(n_0, n_1, \ldots)$ such that

$$n_{p+1} = k_p \cdot n_p + n_{p+1}, \quad \text{where} \quad n_{p+1} \in \{0, 1, \ldots, n_p - 1\}.$$

The obtained sequence (n_p) consists of nonnegative integers and is decreasing, hence it must be finite. The last positive member of this sequence is equal to $GCD(m,n)$.

Here you can find a scheme and program for calculating GCD with application of Euclid's algorithm.

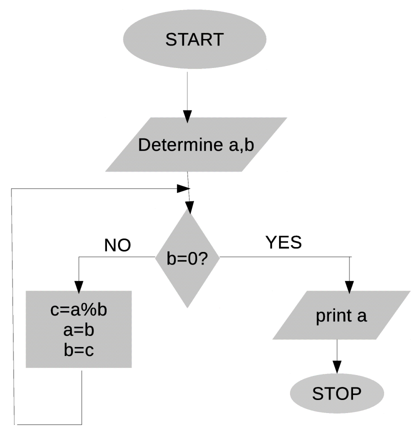

FIGURE 3. **GCD - Euclid's algorithm**

```
EUCLID'S ALGORITHM - PROGRAM IN PYTHON LANGUAGE
print("Determine two positive integers")
a,b=int(input()),int(input())
while b!=0:
   c=a%b
   a=b
   b=c
print("GCD of the above numbers equals",a)
input()
```

5. Perfect numbers, amicable numbers

For pupils which are interested in mathematics, we can enlarge some topics on number theory connected with divisibility. For example, perfect numbers and amicable numbers are very interesting.

Definition 7. *A positive integer n is called a perfect number if it is equal to the sum of all its divisors less than n.*

Perfect numbers state the big mystery in number theory because of:

- We don't know whether the set of perfect numbers is finite or infinite.
- All found perfect numbers, which are known, are even; we don't know if there is any odd one.
- There is no effective algorithm of finding perfect numbers. But there is known a connection between perfect and Mersenne's prime numbers.

First 7 perfect numbers are as follows:
6, 28, 496, 8128, 33550336, 8589869056, 137438691328.

Definition 8. *Two different positive integers are amicable if each of them is the sum of divisors of the other one (less than this one).*

There is a similar problem with amicable numbers. We don't know if there is finite or infinite set of pairs of them and all found amicable pairs are both odd or both even; we don't even know if there is a pair of amicable numbers such that one number is odd and the other one is even.

A few of pairs of amicable numbers are as follows:

$$(220, 284),\ (1184, 1210),\ (2620, 2924),\ (5020, 5564),\ (6232, 6368),$$
$$(10744, 10856),\ (12285, 14595).$$

6. A few exercises and problems for homework

Problem 1. *Calculate (applying Euclide's algorithm) $GCD(240, 600)$ and then $LCM(600, 240)$.*

Problem 2. *Find all divisors of number 1200. Apply the system of writing them in pairs.*

Problem 3. *Find all multiples of 13 which are in the interval $[600, 800]$.*

Solution: $\left\lfloor \frac{600}{13} \right\rfloor = 46$, $\left\lfloor \frac{800}{13} \right\rfloor = 61$. So we have:

$$W_{13[600,800]} = \{47 \cdot 13, 48 \cdot 13, \ldots, 61 \cdot 13\} = \{611, 624, 637, \ldots, 793\}.$$

Problem 4. *Is it true that if a positive integer is divisible by* 4 *and by* 6 *then it is divisible by* 24*?*

One can prove the following theorem:

Theorem 7. *If a positive integer a is divisible by b and it is also divisible by c and additionally $GCD(b, c) = 1$ then a is divisible by $b \cdot c$.*

We can apply this theorem to creating new characteristics about divisibility by composed numbers.

Problem 5.
(1) *How to find principle divisibility of* 6*?*
(2) *How to find principle divisibility of* 24*?*
(3) *How many two-digit numbers are divisible by* 7*?*
(4) *How many three-digit numbers are divisible by* 7*?*
$L = \lfloor \frac{999}{7} \rfloor - \lfloor \frac{99}{7} \rfloor = 142 - 14 = 128.$
(5) *How many two-digit numbers are divisible by* 4 *and by* 6*?*
Since $LCM(4, 6) = 12$, *then* $L = \lfloor \frac{99}{12} \rfloor - \lfloor \frac{9}{12} \rfloor = 8$
(6) *How many two-digit numbers are divisible by* 4 *or by* 6*?*
$L = \lfloor \frac{99}{4} \rfloor - \lfloor \frac{9}{4} \rfloor + \lfloor \frac{99}{6} \rfloor - \lfloor \frac{9}{6} \rfloor - \left(\lfloor \frac{99}{12} \rfloor - \lfloor \frac{9}{12} \rfloor\right) = 24 - 2 + 16 - 1 - 8 = 29.$

How to find the number of divisors?

Problem 6. *Calculate, how many divisors the numbers* 240*,* 125*, and* 3000000 *have.*

Solution: Of course every prime number has only two divisors. If we take into account composed number n we first have to present it as the product of prime numbers:

$$n = \prod_{i=1}^{n} p_i^{k_i}.$$

It can be easily proved that the number of divisors is equal to:

$$L = \prod_{i=1}^{n} (k_i + 1). \tag{1}$$

For example, we want to know how many divisors the number 6000 has. Factorizing the number 6000:

$$6000 = 2^4 \cdot 3^1 \cdot 5^3.$$

Then, on the base of (1) we obtain:

$$L = (4 + 1)(1 + 1)(3 + 1) = 40.$$

Now we can present some simple tasks.

(1) A positive integer divided by 5 gives a remainder 4. What remainder shall we get if we divide square of this number by 5?

(2) First of 2 numbers divided by 6 gives a remainder 4 and the other one divided also by 6 gives a remainder 3. What remainder shall we get if we divide the sum of these numbers by 6 and what remainder shall we get if we divide product of these numbers by 6?
(3) Show some practical tasks in which we use GCD or LCM.
(4) Prove that the sum of two odd numbers is even.
(5) Prove that the product of three consecutive natural numbers is divisible by 6.
(6) Find all pairs of positive integers which are solutions of the following equations:
   (a) $2x + y = 8$
   (b) $3x + 5y = 8$
(7) Find all positive integers n for which the fraction $\frac{2n+16}{n+2}$ is a positive integer.

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USING GRAPHIC DISPLAY CALCULATOR IN SOLVING SOME PROBLEMS WITH POLYNOMIALS

JOANNA JURECZKO

Abstract

A graphic display calculator (GDC) is becoming more and more popular in teaching mathematics as it is used to examine some mathematical activities of students of almost all ages. Various modes of GDC are considered to be a useful tool in understanding of particular parts of mathematics. In most cases the properties of functions are examined by observation of their graphs. However, there are some properties of the functions which one cannot see during the graphs analysis (for example properties concerning complex roots of polynomials). The aim of this paper is to analyse how 17-and-18-year-old students for whom GDC is an obligatory device can generalize some relations between polynomials and so called "shadows" of these functions. The whole paper is concerned with in investigation of properties of quadratic, cubic and quartic functions with both real and complex roots.

1. Introduction

A graphic display calculator (GDC) has become more and more popular in the process of learning and teaching mathematics. There are lots of researches about an effective usage of this portable device (for instance [1], [2], [3]). Some of them show how students use GDC as a routine activity for them (for instance [4], [5]) other show teacher's expectations of using this tool (for instance [6], [13]). However, there is no lack of papers which show difficulties in using GDC (for instance [14]). In Polish school programs graphic calculator is not popular. Moreover any calculator, except for simple one (four-operations one), is forbidden during Polish exams on each level in both the middle and high schools. The unique school programme admitted by Polish Minister of Education – so called International Baccalaureate Diploma Programme – accepts the use of GDC as a mandatory device during the process of learning – teaching mathematics and during the public examinations. It is alternative programme in high schools and

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it is intended for students aged 16-19. (For further information about this program one can go to [11], [12], or websites). As using GDC is not present among Polish maths teachers. In [7], [8], [9], [10] I described my research with graphic calculator was carried out among my students attending International Baccalaureate Diploma Program class (shortly IB class). For my research I mainly chose tasks or parts of tasks intended for using IT (especially graphic calculator). However, other tasks I also used. Students solved the tasks during the normal lesson time or had comfortable conditions – they worked prior to the lesson time, individually without time limitation, having access to GDC, the Internet and computer software all the time. Unfortunately, we have no chance of knowing how the quality of conditions during solving the tasks influenced students' work (activity performed on GDC and computer was not recorded).

The main role in this paper plays the task patterned on a task proposed in "Portfolio tasks for use in 2012 and 2013 published by International Baccalaureate Organization" (more about Portfolio one can find in [11], [12]).

The current task which analysis will be considered in this paper concentrates on the investigation of some properties of polynomials.

As authors emphasized in [14] a graph of function is a crucial weapon in the mathematics learning. As early as possible students learn to recognize the important features of graphs of functions. They are taught how to find intercepts, roots, monotonicity of such functions like linear and quadratic functions. Yet, when GDC is not known or available the numbers of graphs that students can draw is rather limited. As a consequence students usually have a problem with finding or changing the scale if they use only paper and pencil for this purpose. If they have to find some properties of graphs they usually sketch graphs of "nice" functions, for instance with roots or vertices which are integers. However only after students use any technologies for such tasks they become real researchers. They can investigate many examples with different properties (different kinds and numbers of roots, etc.) Proposed task in this paper has some signs of the process of generalization. This process is considered by me in [10], where I examined so called visual templates (name proposed by Rivera in [15]). Learning on my research and remarks made in [15] I proposed the scheme presenting the process of generalization using such a special kind of tasks.

This scheme was verified in [7]. Below the scheme is quoted, (see Table 1).

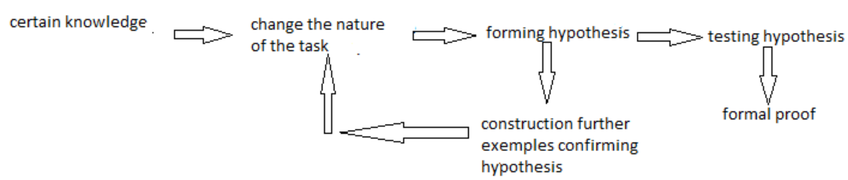


Table 1. The scheme of process of generalization proposed in [10]

In paper [3], which is worth mentioning here, one can find division of applications of GDC during almost all mathematical activities.

1. Computational Tool (evaluating a numeral expressions, estimating and rounding)
2. Transformational Tool (changing the nature of the task)
3. Data Collection and Analysis Tool (gathering data, controlling phenomena, finding patterns)
4. Visualizing Tool (finding symbolic functions, displaying data, interpreting data, solving equations)
5. Checking Tool (confirming conjectures, understanding multiple symbolic forms)

The main questions which I asked prior to the research were

1. How useful can a graphic calculator be in solving the task with different kinds of polynomials?
2. Can students observe common properties of roots using only a graphic calculator?
3. Can students solve such tasks without any technologies?

2. Main results

The research was conducted in the class with International Baccalaureate Diploma Programme. All students who took part in the research (ten 17-18-year-old students (eight boys and two girls) were taught by me. The whole research was concerned with one task patterned on the portfolio tasks for mathematics in IB programme for use in 2012-2013 titled "Shadow functions" published by International Baccalaureate Organization (IBO). Below one can find the text of the task.

The task

Consider the quadratic function $y_1 = (x-a)^2 + b^2$. *Write down the coordinates of the vertex and show that all roots of* y_1 *have the form* $a \pm ib$ *where* $i^2 = -1$. *Consider the function* y_2 *which has opposite concavity to* y_1. *Such function is so called "shadow function". Use various values for* a, b *to* *generate pairs of shadow functions. Let y_m be a line of reflection of y_1 and y_2. Find equations for y_1 and y_m. Express y_2 in terms of y_1 and y_m. On the diagram show how zeros from y_2 may be helpful in determination of roots of y_1. Consider the function $y_1 = (x+2)(x-(3+2i))(x-(3-2i))$. Let y_2 be a shadow function for y_1 which has one common zero at point -2. Repeat all points above mentioned. Check whether your results can be applied to quartics.*

This task is concerned with the observation of pairs of functions (named shadow functions) which have different roots but the graphs have reflected shapes in some lines. This task seems to be created for using GDC because students have opportunity to examine not only polynomials with real roots but also with complex roots.

One should know that prior to this research students were taught about roots of polynomials and about complex numbers Nevertheless, they did not solve such formulated tasks and did not generalize similar problems.

The research was divided in three parts. The first one was the observation of students working on the task during three consecutive 45-minute lessons. Because the task consisted of three similar parts I expected students to solve at least the first part of it (about the quadratic functions). At the beginning of the lesson the students were given the task and graphic display calculators (one copy of the task and one graphic calculator per student) and explained very carefully the problem included in the task. However, at this point the students did not obtain any particular hints. During the lesson time students worked with the task individually and wrote their solutions on the provided sheets of paper. Throughout the task I only observed students. When the time was over I gathered the sheets in order to analyse them.

The second part was the interview which was recorded. The students were asked individually the same five questions quoted below

1. How did you use GDC for solving this task (which mode of GDC did you use and why)?
2. Does GDC is a sufficient device to solve this task. What else do you need?
3. Did GDC let you generalize the problem from the task? How?
4. Can you solve this task without using any technologies?
5. Having GDC would you like to formulate any kind of task for your peers? What kind of task would it be?

In the third part of this research I provided the students the same task and made them to solve it individually during 10-day-period as their homework. After this time I gathered students final works in order to analyse them.

The analysis of students' work

As each part is considered to be a different part of an activity I will analyse them separately.

The first part concentrates on the analysis of students notes made during three consecutive lessons. Generally, students solved only the first part of the task (about the quadratic functions) because the time was limited and did not allow them to do as many attempts of examinations of similar examples as were needed. Despite the limited time, some students worked quicker and tried to check obtained generalization for cubic functions. However, they did not finish their work, so the analysis of the cubic functions is limited. It is important to notice that six students used GDC but others did not do it. Students who used GDC usually used it for sketching graphs of functions given in the task and other similar examples produced by them. Moreover, they checked other properties of the graphs (x- and y-intercepts, roots, etc.) As a result using mentioned above properties they draw some relations between roots of both functions (functions and their shadows). Below the original work of three students is presented below, in which they included generalizations between roots of y_1 and y_2 (compare the text of the task).

Student 1 found general patterns for roots of y_1 using adequate patterns for given quadratic function. Next he wrote the pattern for y_2 and examined three examples for different values of a and b, but only for a, b being integers. In the first one he found roots for both y_1 and y_2 using adequate patterns but in further examples he found roots using GDC. He finally wrote conclusion but only for the first example proposed by him, i.e. for $a = 2$ and $b = 1$. What is important, he did not checked whether his conclusion was suitable for further examples. Below the original piece of work of student 1 is provided, (see Table 2).

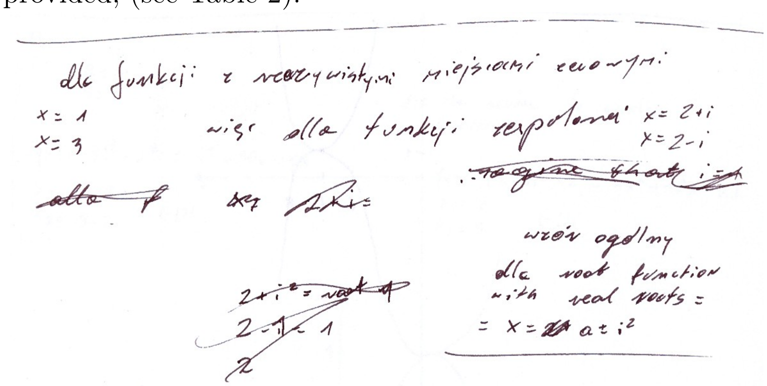


Table 2. A part of original student's work (student 1)

for the function with real roots $x = 1, x = 3$, *so for complex functions* $x = 2+i, x = 2-i$. *General term for root function with real roots* $x = a \pm i^2$ *(trans. by J. Jureczko).*

This pattern cannot be recognized as general pattern. The student should have checked his pattern for other examples but he did not do it.

According to the task he gave the pattern for y_m and expressed y_2 in terms of y_1 and y_m using only general patterns for y_1 and y_2 and solving the following system of equations

$$\begin{cases} y_1 = (x+a)^2 + b^2 \\ y_2 = -(x+a)^2 + b^2 \end{cases}$$

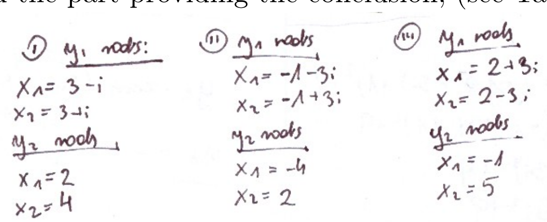
where $y_m = b^2$.

Following the instructions student 2 did not find roots of y_1 but only checked that they really were by substitution consequently obtaining tautology. She found the pattern for y_2 properly but did not formulate the general pattern (only used in investigated examples). Then she examined three examples for different values of a and b, in each case checking algebraically whether complex roots were really roots of given function. Yet, for roots of y_2 she did not do it. For all examples she found roots using GDC.

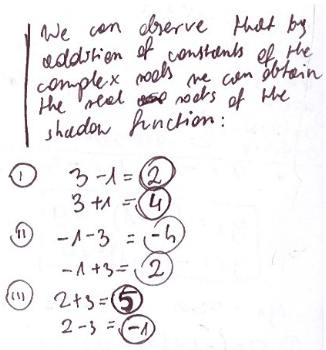
Following the further points of the task she found the pattern for y_m. Nevertheless, in order to express y_2 using only terms y_1 and y_m she incorrectly assumed that $y_2 = -y_1$ and then she wrote the pattern for $-y_1$ twice and crossed it. In spite of this mistake she used proper patterns for y_1 and y_2. By writing in the next line

$$(x-a)^2 + y_m = -(-(x-a)^2 + y_m) + 2y_m$$

and by reducing the similar terms she obtained $y_2 = -y_1 + 2y_m$ which contradicted her assumptions. Additionally, she did not comment on this situation. Afterwards it she tried to find the relations between roots of both quadratic functions. The first example if for $a = 3, b = -1$, the second one is for $a = -1, b = -3$ and the last one is for $a = 2, b = 3$. Below one can find the part providing the conclusion, (see Table 3.)



i	ii	iii
y_1 roots:	y_1 roots	y_1 roots
$x_1 = 3-i$	$x_1 = -1-3i$	$x_1 = 2+3i$
$x_2 = 3+i$	$x_2 = -1+3i$	$x_2 = 2-3i$
y_2 roots	y_2 roots	y_2 roots
$x_1 = 2$	$x_1 = -4$	$x_1 = -1$
$x_2 = 4$	$x_2 = 2$	$x_2 = 5$



We can observe that by addition of constants of the complex roots we can obtain the real roots of the shadow function:

(i) $3-1=2$
$3+1=4$

(ii) $-1-3=-4$
$-1+3=2$

(iii) $2+3=5$
$2-3=-1$

Table 3. A part of original student's work (student 2)

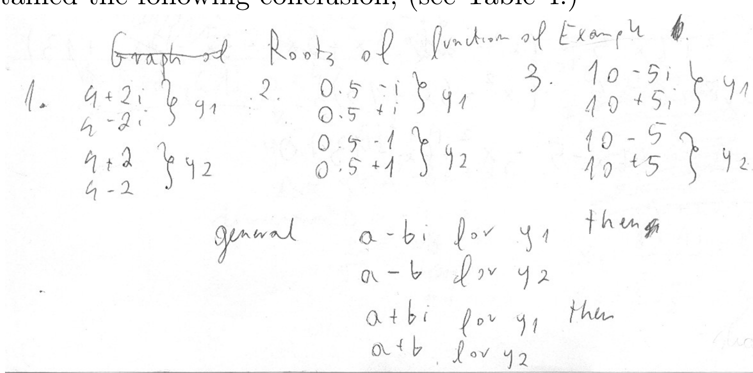
In comparison to student 1 student 2 did not write the general pattern algebraically. She only wrote verbally how one could obtain roots for arbitrary shadow functions. Moreover, she did not examine further examples to confirm her conclusion. Even though she tried to use her conclusions shown above for given cubic function, she did not check it for any examples of the cubic function.

Student 3 found the roots of y_1 using the same method as student 1 and gave the pattern for y_2 properly. Then he examined three examples for different pairs of values of a and b by finding roots using paper-pencil method and checking with GDC. Although he used proper patterns for y_1 and y_2 he concluded general expression for y_2 in the form

$$y_2 = -(y_1 - y_m)^2 + y_m.$$

He made a mistake as he substituted the expression for $(x-a)$ by $y_1 - y_m$ omitting the square power in the first expression.

For the values: $(a = 4, b = 2)$, $(a = 0.5, b = 1)$ and $(a = 10, b = 5)$ he obtained the following conclusion, (see Table 4.)



~~Graph of~~ Roots of function of Example

1. $4+2i$, $4-2i$ } y_1; $4+2$, $4-2$ } y_2

2. $0.5-i$, $0.5+i$ } y_1; $0.5-1$, $0.5+1$ } y_2

3. $10-5i$, $10+5i$ } y_1; $10-5$, $10+5$ } y_2

general $a-bi$ for y_1 then
$a-b$ for y_2
$a+bi$ for y_1 then
$a+b$ for y_2

Table 4. A part of original student's work (student 3)

Student 3 provided a general pattern for roots of shadow functions algebraically. Similarly to student 1 and student 2 he did not check his general pattern by examining further examples.

As student 2 he tried to apply his conclusion for given cubic function. Yet, he did not check it for any cubic function.

Three other students tried to obtain the similar results. Although they had a good idea, the time probably did not allow them to finish them this part of the task. Other students, who did not use GDC in solving the task, operated only on the general patterns (using letters instead of numerical examples of functions), but none of them obtained neither relations between the graphs showing on the task nor other properties mentioned in the task.

The next part of the research was an interview with participants which was done immediately after these three lessons. Below there are provided citations of only those three students whose solutions were analysed above (the transcript from Polish is translated by me)

Answers for question 1:

1. Using it I could find roots faster. I could sketch graphs and check different kinds of functions. (I used) Graph, Equation, Run Math.

2. GDC helped me in sketching graphs which was needed to solve the task and to obtain roots. It also made the task easier and helped me to get a solution faster. (I used) Graph.

3. I checked my answers whether they were proper and sketched graphs of functions I tried to compare this functions with complex numbers but I did not manage to do it because the time was too short.

Comments: Students claimed that they used GDC generally as a visualizing and checking tool (compare [3])

Answers for question 2:

1. This device is sufficient to solve this task because it has all needed options to do it.

2. I think this device suffices but if I want to prepare it better I would need Geogebra to sketch graphs.

3. If this task was precisely done I would need to use computer to sketch graphs.

Comments: Students claimed that they did not need any other devices as far as they had to prepare their work very carefully. They understood that the screen of GDC an its software was as not precise as computer software.

Answers for question 3:

1. GDC helped me to generalize the problem posted in the task and helped to make conclusions

2. No, rather not.

3. Rather not, I wrote on the piece of paper and used GDC only for calculations and graphs.

Comments: Students claimed that GDC helped them only for checking a few examples (as a transformational tool), but for generalization human logical thinking is needed. As a consequence only student 1 gave positive answer, but he did not obtain generalization required in the task properly.

Answers for question 4:

1. I would be able to solve this task without using GDC but it would take me more time.

2. I think so, but it would be more workable.

3. It would take more time.

Comments: Students unanimously claimed that they were able to solve the task without any technologies but it would take more time.

Answers for question 5:

1. No, I cannot formulate any tasks because I have no idea how to do it.

2. If I created some task it would be the task of type with functions because GDC is more useful in sketching graphs and finding their properties.

3. I would try, it would be interesting.

Comments: Students had no precise idea how to generate the new task probably because they did not have enough experience (this task was the first task of this type for them).

The last part of the research concentrate on the analysis students' work which was done during 10-day period as their homework (without time limitations and with the full access to GDC and computer software).

During the 10-day-period almost all students solved the task correctly, the method of thinking which started during the lesson time was continued by them. What is worth emphasizing all students used GDC and graphic computer software to solve the task. Although this part of the research was the most progressive as the students solved the task without time limitation and with comfortable conditions the researcher does not know about anything about the attempts of solving the task, the process of reasoning, methods of work or even the time needed for solving the task. Without any students' explanations it is difficult to analyse the process of solving the task.

3. Final remarks

To summarize that students solving the task worked under two different conditions. The first one was during the three consecutive lessons which were during one day. In this part of research time for solving the task was limited. Students who used GDC for this task mostly obtained a part of required solution (especially for quadratic function).

What is important to notice: the parts where they used GDC were without any mistakes, but when students had to conclude some general patterns they made mistakes in reasoning (student 2 and 3) and other calculations (student 3). If they had worked without any technology they probably would have made more mistakes and the general statement about roots of functions would be impossible. What is strange students, for producing examples of functions, used only integers or simple fractions instead of constants a and b. Some students (student 3) preferred to calculate roots without GDC and only checked the result. Students probably were afraid of not obtaining a precise solution and they were right because for other constants they could not perceive required relations.

Generally, students used GDC for sketching graphs, checking results and some simple calculations. As was mentioned in paper [3] they used it in almost all way except for "Data Collection and Analysis Tool", but they did not use it during learning, so this may be the reason for this situation.

What is worth pointing out students in this task omitted two steps proposed in my scheme. After making hyphotesis they did not try to do any further examples to confirm it. It followed some dangers of incorrect general pattern (see work of student 1). They did not do any formal proof for confirming their generalized patterns although they were able to do it.(It was mentioned in the paper [10] too).

After analysis of the research, one can make the conclusion that the role of GDC is double: to form hypothesis (by examining a big number of similar examples made using GDC) and to formulate the general patterns.

In spite of some problems with using GDC it is still worth analysing students' work, especially in order to find the role of GDC in process of working on more complicated tasks, because if one has GDC or other computer software one can draw a graph of each function, check its main properties even if the roots are not integers or rationals. If a student has technological devices one can make a number of attempts in short time which make it easier to observe similarities or differences and conclude generalizations of observed objects.

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WORKSHOPS AS A SIGNIFICANT VOCATIONAL EDUCATION COMPONENT OF STUDENTS SPECIALIZING IN TEACHING FIELDS

MARIA KORCZ

Abstract

The post modern social processes have somehow destroyed the institutional role of teachers over the past two decades. It is currently necessary to give would-be teachers new, different competencies than the present ones. The tendency to make the teacher education adequate for contemporary needs imposes a number of new tasks on the institutions of teacher education. There is, among other things, a need for organizing innovative classes in terms of both their content and form. The article will contain a practically verified concept of workshops as an important form of improving and completing the education of mathematics teachers.

1. Introduction

The post modern social processes and the withdrawal of Herbart's pedagogical concepts in favour of holistic pedagogy have significantly changed the perception of a teacher's role over the past two decades. It is currently necessary to give teachers new competencies other than the present ones. Their content of the competencies should not be highly specialized but joint, not closed but open, not reproductive but more creative. School teachers should tend to be guides and explainators rather than executors. There are ongoing discussions on the methods of teacher education and its desired effects in numerous countries. In 2002 the Ministers of Education of the European Union member states and the European Commission approved a 10-year program of education development whose first specific objectives among 3 strategic objectives is to improve the quality of educating teachers and in-service training for them. In 2005 the activities of a work-group constituted by the European committee were summarized. The work-group consisted of the representatives of 25 countries and developed a competency packet of a European teacher. The competencies were divided into the three

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following fields: learning organization, education the attitudes of school children and incorporating extra subject competencies to the teaching content of a specific subject.

Firstly, it is remarkable that the term "competencies" is often confused with a scope of the knowledge about any specific subject. Yet, the knowledge is merely one of the factors that determine the acquisition of certain competencies. It is highlighted by Weinert's definition of competencies. He defined competencies as one's cognitive skills and abilities to solve problems with the motivational and social readiness to use them in order to solve various problems in an effective and responsible way in ever changing situations. Secondly, the 21st-century schooling is not understood as as encyclopedic teaching, critical thinking, discussing and analysing but, above all, as acting. It can be expressed, in other words, as thinking for the sake of acting. Therefore, the tendency to adapt teacher education to contemporary needs results in numerous new tasks and changes in a number of teaching fields. The tasks are to be performed by the institutions of educating teachers and providing them in-service training. The education system favoured by high schools and universities is still based on conveying knowledge to students. Lectures and academic classes are at the core of the system. Such teaching methods do not support their students' activities and make it difficult for them to obtain appropriate pedagogical competencies. There is an urgent need for organizing innovative classes in terms of their form and content.

2. Workshops as an innovative apprenticeship form

The project "Practice makes perfect" was carried out at the Faculty of Mathematics and Information Science UAM from 2010 to 2013. The project was intended to develop a model of the teaching apprenticeship designed for the mathematics students who are prospective mathematics teachers. Workshops were an essential novelty in its program. The workshops were organized as several-day field classes. Most students had never known such form of classes before. The subject discussed in the workshops were divided into two groups:

(1) methodological problems concerning mathematics and information science and

(2) psychological and pedagogical problems. On one hand, the problems from the first group were determined by staff availability (both university employees and non-academic teachers). On the other hand, the problems were selected on the basis of the need of the participants to thoroughly explore certain topic areas. Both ways of selecting the topics can be considered as peculiar because of the participants' own needs. Their contents were basically included in the curriculum of mathematics didactics. Thus,

it seems to be pointless to present the detailed program of the first group of problems in this article. Nevertheless, the psychological and pedagogical group of problems was innovative in terms of its form and the content of its component classes. Therefore, it will be discussed in more details.

The contents of the group were not related to teaching psychology which is typical of course lectures on psychology. This part of the workshops was devoted not to direct improving the teaching process but to developing the students' personalities.

The following problems were raised within the psychological–pedagogical workshops:

Training interpersonal skills.

Emotions in interpersonal relations.

Assertive techniques.

Time and teaching resources management.

Planning an educational work, working with a student group and its surroundings.

Reasons for choosing the job of a teacher.

Content-related and the teacher's psychological competencies.

Practical communication aspects and the development of the teacher's social competencies.

The problems of communication contexts. Utterance vs. Information. Received utterance. The problems of the multi-dimensionality of the utterance listening process.

The art of communicating.

The art of maintaining constructive relations in the education process.

Managing the communication process in the education process.

Management styles and school practice.

Conflict as a causality in the teacher's work.

Dyscalculia and its relations to mathematics teaching.

It is remarkable that every teacher should have the above competencies and they are independent on the characteristics of different subjects. The teacher ought to be able to develop their own personality as a structure of skills and values. They will allow them to find a solution to various pedagogical tasks and problems which are typical of their job. It is a truism to say that the teacher's job is particularly stressful and demanding. It is caused by its social mission and as a consequence, its responsibility and expectations related to its performance. The necessity to maintain a stable and difficult contact with other people is a challenge. In complicated interpersonal relations in schooling teachers quickly confront not only with certain limitations of the education system but also with their own confinements. To cope with the challenging tasks university students need to have classes aimed at developing practical skills with an emphasis on psychosocial competencies. The latter ones are presently neglected in high school teaching. The students had an opportunity to assess the usefulness of the suggested workshops. These are the pieces of their opinions: The workshops played a crucial role in the project. Thanks to the chance to take part in the workshops we know much more that our co-students who did not take part in the project. We really learned a lot. It is impossible to express it briefly. We spent the entire days on the classes, conversations, learning activities, sharing our experiences at nights, solving problems in the passages, endless conversations or sincere dialogues with the school teachers. The psychological and pedagogical workshops allowed not only to get to know oneself better but to make us realize our role in the workshop community. It might help us settle in at school as a social group. The workshops had an impact on my idea of working as a teacher.

We are going there to start working hard. In this part of the project many topics were raised and precisely discussed. It seems to me we cannot do all these things in our regular classes during the academic year despite all the efforts of the university authorities. Despite the theoretical part I would draw your attention to the psychological and pedagogical classes which helped all the students get to know each other much better. We enriched our practical knowledge by sharing our experiences during the consecutive meetings. Beginning teachers gain such knowledge in the first few years of their work. We might discuss all the problems and think them over thanks to the complicated questions in workshop meetings... One might ask a question: Do students have identical chances in their standard curricula?

The workshop form was very interesting. We could learn a lot of new things which we had not known before. Personally, I got most useful information from the psychological classes. During the workshops we had a chance to all the didactic problems from the school teachers' point of view. We could learn how to cope with various situations.

3. Conclusion

The organization of classes as workshops is the only effective method to verify one's own relations with other people and to cope with their possible results. In turn, it is an appropriate point of departure to start searching for solutions to various problems during the workshops based on the knowledge, experience and support of other group members.

The students have often declarative knowledge about their social skills. The students happened to reveal impatience in the workshops when a topic previously discussed in psychology or pedagogy classes was raised. But if an instructor skipped "the theoretical preliminaries" and moved on to the tasks on the practical use of the theory, it turned out very quickly that the students did not have such skills at all.

Before and after the workshop sessions the students underwent a survey research on their self-assessment of their psychosocial competencies necessary in the teacher's work. The survey included the questions about such competencies as: describing their pedagogical resources and personal advantages useful in the teacher's work, conscious creation of their image, recognizing and naming one's own emotions, communicative competencies, emphasizing in other people's problems, solving conflicts, collaborative skills, managing a group work, handling with one's tension, adapting to the changes in the surrounding. The comparison of the survey results before and after the workshop showed that students tended to overestimate their practical skills and confused them with the theoretical knowledge obtained at the university. The confrontation of the theory and the praxis raised the student participants' awareness of their lack of practical competencies. In extreme cases, some students were in the lower self-assessment. These students claimed that they learned much during the workshop. They admitted having overestimated their competencies and were not simple conscious of their practice knowledge shortage.

Despite the teaching aspect, the key role of the workshops is to integrate their participants. The talks during workshop sessions allow to get acquainted with their participants' needs and the possibilities to commence common actions by the school and university communities.

To sum up, the role of the workshops as part of the preparation process of pedagogy-oriented students to work at school in very valuable. The workshops are an essential completion of the process with an opportunity to confront the teaching theory with its practical aspects, particularly within the psychosocial competencies. Such elements were missing or included in the present university curricula to minor extent.

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CAN INFORMATION TECHNOLOGY CHANGE ATTITUDE OF THE STUDENT IN THE TEACHING OF MATHEMATICS?

JAROSŁAW KOWALSKI

Abstract

The article shows a specific example of extracurricular activities conducted in secondary school, how the graphing calculator helped the first class student in learning mathematics to solve a very difficult task (math problem): *How many elements has the equation*: $a^x = \log_a x$. The article describes the reasoning and attitude of the student who voluntarily of his own accord, inspired by other students to experiment, putting, generalizing and verifying hypotheses coped with the solution of this task. It describes the impact of this teaching mean that triggered activity and aroused student's interest with the task on the degree of knowledge and skills in mathematics, the student's skills in the use of mathematical language, self-reliance in solving a mathematical problem.

1. Introduction

Currently in the teaching and learning of mathematics increasingly used information technology (computers, graphing calculators, tablets) noting the great advantages of these teaching aids. Using them helps students to understand mathematical concepts, provides an opportunity to explore the claims, provoking hypotheses and results that the student is able to solve the unusual task, which without this tool on your level of knowledge and skills would not be able to cope. However, there are doubts whether after using a computer or a graphing calculator series of empirical tests and putting the hypothesis, the student will want to even verify and justify them? This raises the question whether the use of this tool does not fail the proportionality between the empirical inference and reasoning formal? Teachers today are wondering: what the consequences might bring the teaching of mathematics, in which, a teaching aid will permanently be woven into your

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computer, tablet or graphing calculator? For me, it is particularly interesting question: Can the continued use of information technology change the attitude of the student in the teaching of mathematics?

The term attitude has a lot of meanings and it is difficult to clarify or even define the term. Anna K. Żeromska [2004] in the article: "The attitude of the conceptual category on the example of attitudes towards mathematical tasks". Cited several terms of definition of the term, commented on the differences between them and considered the definition of the main features that make up each posture. Here are a few examples cited by her attitude definitional terms: "*Attitude – this is the probability of occurrence of a particular behaviour in a given situation*" – W. M. Fuson (Mika, 1982, p. 112).

"*Attitude – this is the amount of the positive or negative feelings associated with some object*" – L. L. Thurston (Mika, 1982, p. 113).

"*Attitude – a structure composed of cognitive elements, i.e. the set of beliefs concerning the possibility of implementing some of the values of the subject posture and affective feelings caused by this item*", (Rosenberg, 1962).

"*Attitude – is formed in the process of meeting needs in specific social conditions, relatively stable organization: knowledge, beliefs, feelings, motives, certain forms of behaviour and entity expressive reactions associated with a particular object or class of objects*" (Mądrzycki, 1977, p.18).

According to Żeromska the concept of attitude can point to some elements repeated in most of the theoretical thinking and forming a common core of meaning of the term. In the concept of attitude, there are three interrelated components:

(1) cognitive – meaning the overall knowledge and beliefs relating to the subject
(2) emotional – including feelings about a particular subject
(3) behavioural – relating to behaviour, reactions and actions aimed at target responding to the question: what is the attitude of the body relative to the object?

The study psychological M. Morady [1976] trying to create a tool for measuring and evaluating, testing attitudes. What should examine, measure, observe to infer about the attitude. The term attitude is also used in the teaching of mathematics to describe and explain the behaviour of the student in situations related to the teaching and learning of the subject, e.g. M. Legutko [1984], M. Czajkowska [2002, 2003], M. Ciosek [1976, 1988, 1992, 1995]. The teaching of mathematics also explores issues of students' attitudes towards same mathematics e.g. Ruffell, Mason, Allen [1998]. I would like to examine how forms, grows and changes the attitude of the student under the influence of such a factor as the continued use of information technology in teaching and learning mathematics. My researching has a long lasting character. They have to investigate and describe the impact of the constant use of TI (graphing calculators) in the teaching and learning of mathematics in shaping the attitudes of students towards mathematical tasks which means checking pupils' progress in science and showing the results of learning (taking into account all three components of attitudes that manifest themselves in any posture, but in different proportions).

In Contemporary Problems of Teaching Mathematics in the article: "*The role of graphing calculators in solving mathematical problems*" I have presented a specific example conducted lessons in grammar school, how a graphing calculator can help solve very difficult for middle school students the task (a mathematical problem): How many elements has the equation: $a^x = \log_a x$. In this article, I described the reasoning of students occasionally using information technology in the classroom mathematics who inspired to experiment, putting, generalizing and verifying hypotheses coped with the solution of tasks using a graphing calculator. In this article, I have attempted to describe the attitude the student (applying IT in the learning of mathematics) to the contents of the attitude this above math task is examined.

2. Description of experiment

Description of the experiment:

For extracurricular activities reported to me first class student who wanted to try to solve this problem task. Due to the different way of solving that has not occurred in the work of older pupils with whom I conducted an experiment is worth presenting. The student – Nathan math lessons program made at a basic level and was familiar by the day of solving the problem with the following material: real numbers, the function and its properties, linear function, vectors, transformation of graphs of functions, quadratic function (without the task of parameter) trigonometry (acute angle). Nathan is a student who frequently uses technological innovations. He can take advantage of: laptop, tablet, and smartphone-pad for presentation his solution, explain the present problem, its visualization. He often uses i-pad at school and has installed a number of mathematical applications that are used in solving the tasks of school. Attends extracurricular activities, taking part in the project: "Development and implementation of compensatory course in mathematics using ICT for secondary school students" project entitled "Mathematics Re@ktywacja" within the Operational Programme of the European Social Fund. The initiators and executors of the project are employees of Wrocław University of Technology and coordinator of the project is Ph.D. Jędrzej Wierzejewski of the Institute of

Mathematics and Informatics, Wrocław University of Technology. "Mathematics Re@ktywacja" uses a support system of teaching, in which advanced technologies are closely linked with the active participation of teachers and students. Students receive through the Internet remote access to comprehensive lecture materials and a vast number of dynamic interactive exercises and tests of the entire range of mathematics covered in the curriculum in secondary schools. Lecture materials and exercises are intertwined in such a way that the student getting to know a new issue or a method could immediately proceed to independently solve problems. Project: "Mathematics Re@ktywacja" complements the academic program for secondary schools in the field of interactive math remedial classes. Material e-learning can be a substitute for lessons, but its principal advantage is the possibility of individualized pace of acquiring knowledge, practising multiple tasks in different sets.

In an interview with the student I learned that he had heard from older friends from school about an interesting task which solved using a graphing calculator. This student without any pressure from outside, without coercion, without waiting for the reward of his own accord showed great desire and commitment to try to cope with the task. It was evident in his positive attitude to the experiment, interest, intrinsic motivation to solve this task.

Before attempting to solve the task the student was asked for news of graphs and properties of exponential and logarithmic function. He stated that all the messages he met while working alone on the platform: "Mathematics Re@ktywacja". He made the platform: e-test with exponential functions and logarithmic at 92% scored 46 points out of 50. In an interview said that the unusual task solved by using a calculator or a computer. He had sufficient mathematical knowledge and fluent ability to use graphing calculator TI-83.

The research material only represent the student's notes-draft and completed card work on the task. Running classes not taped because the classes were not pre-planned and I was not prepared for it, however my conclusions, observations of the conversation with the student are provided in the text below.

I introduced the student to solve the problem:

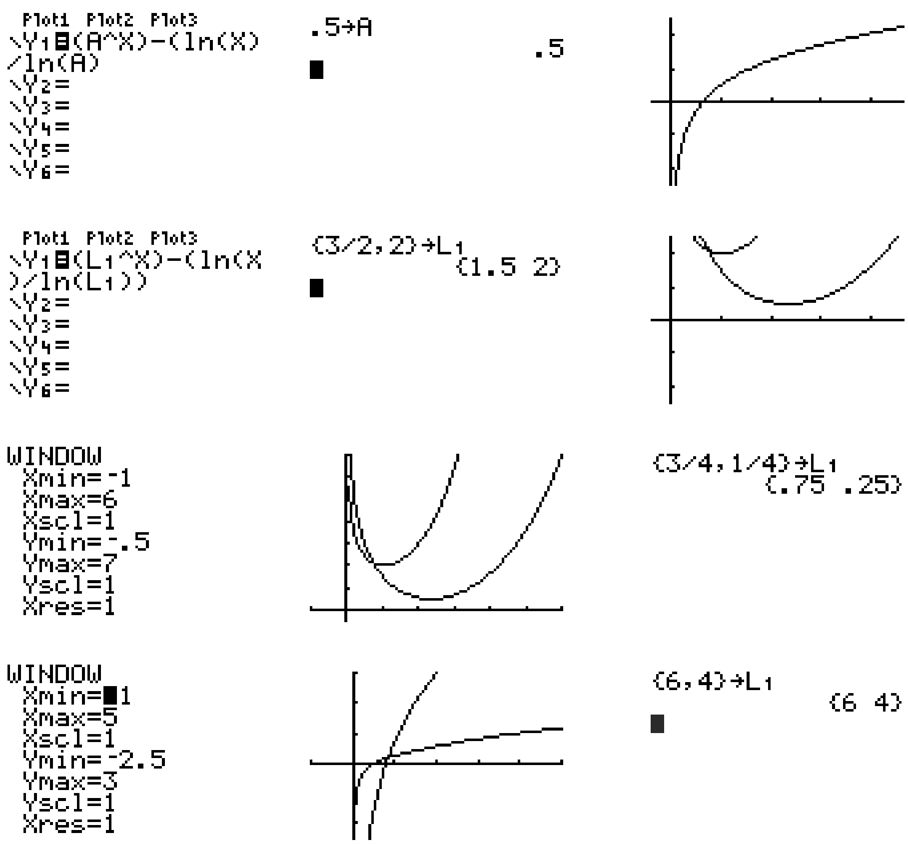
How many elements has the equation: $a^x = \log_a x$.

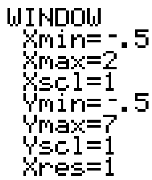
Presenting the draft and work card, I asked to publish his proposals for solutions, observations, notes, charts, and left him with the problem. Nathan asked for a graphing calculator. For a long time I did not pay attention to his work. I watched only his gestures, facial expressions, glances, I noticed only that he was working on a calculator and drew notes.

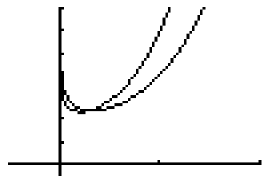
Then I looked at the note prepared by him, and in the conversation I asked for explanation of the records. Nathan explained that the properties from the equation functions: exponential and logarithmic, you can specify the domain of the equation $x > 0$ and the value range of the parameter $a \in (0;1) \cup (1;+\infty)$. He could not solve such equation algebraically at the level of his knowledge . The task of this type meets the first time. This task seems to him to be very difficult to solve.

He took, however, the following strategy:
He moved everything to one side, turning the output equation for the equation of the form: $a^x - \log_a x = 0$.

On the left side of the equation given functions: $f(x) = a^x - \log_a x$. He explained $f(x) = 0$ means that the zero of the function $f(x)$ is the root of the equation: $a^x = \log_a x$. Thus, the number of elements of our equation is equal to the number of zeros of functions $f(x)$ and the number of zeros of functions $f(x)$ can be read from the graph. Reported for selected values of a, the following graphs on the calculator:



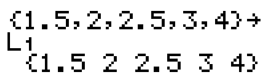


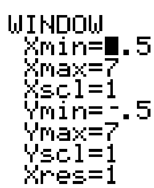


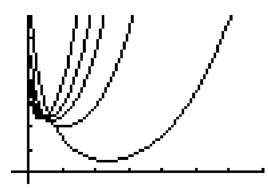
On the job card moved graphs drawn from the Calculator functions.

Nathan explained that in order to solve tasks considered cases for fixed values of the parameters a and based on observations formulated by the above hypothesis. Without giving proper solutions jointly analysed again drawn up by the charts. Discussion on the choice of the parameter value a and deeper analysis prepared charts and look at the shape of the curve being the graph of a function has raised doubts for the student about the hypothesis. Intuition caused reanalysis and observation and the shape of chart dictated to, that might be two zeros. Nathan looked at the field of functions $x > 0$ once again, as well as the range of values of the parameter $a \in (0;1) \cup (1;+\infty)$. Convinced of the need to verify the hypothesis pondering the solution of the task, planned that must be more to look at the graphs of functions for value of the parameter a in the various ranges $a \in (0;1), a \in (1;+\infty)$. This demonstrates the ability of the student to observe, analyse, process and use of the information received. He decided to read from prepared charts the number of zero will be applied to the axis of the numerical values of this parameter a.

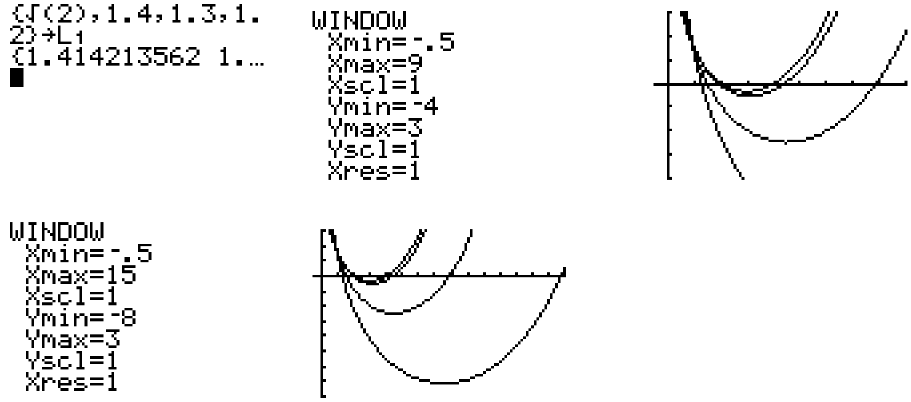
He began work from the value of the parameter: $a \in (1;+\infty)$



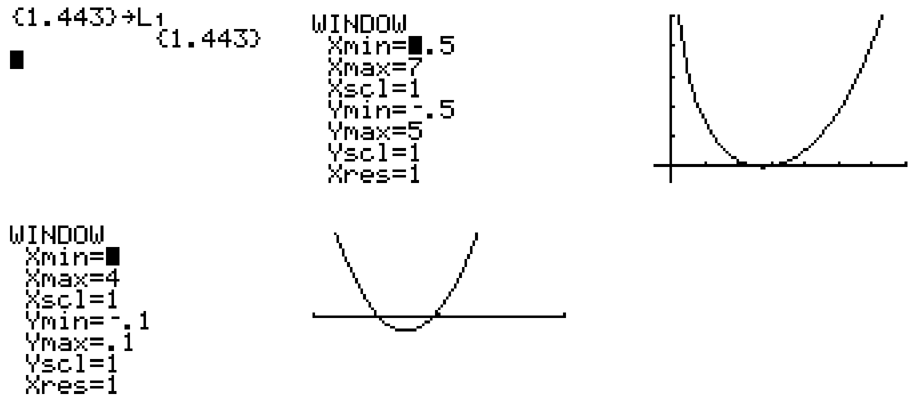




Under the influence of observations drawn graphs of functions on the calculator screen noted that when: $a \geq 1,5$ there are no number of zero.
Next came the observation charts for: $1 < a \leq 1,5$
Working function graphs showed that in this range can be found a value for which there are one or two zeros.



Observation charts led him to the conclusion that in the interval $(1,4;1,5)$ it must be such and such value for a function: for example $a = B$ where function: $f(x)$ has one number of zero and for $a \in (1;B)$ there will be two zero seats. I asked him for the appointment of the value of the parameter a.



In this graphic, method Nathan saw a chance to search, narrowing, bisecting the interval searches. Throughout the work critically approached the information read from the windows calculator. By varying several times of parameters of the chart window, yielded to verify his results. He noted that these attempts fail and determine the exact same value graphically even using a calculator is not possible however in his work was very persistent. He marked it by B and estimated that approximately is: $1,444$.

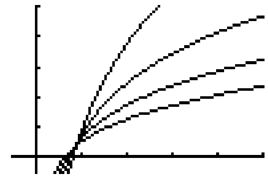
On the job card moved graphs of the Calculator window:

On the axis number for the parameter a he noted the number of zeros and wrote a proposal:

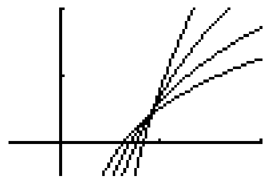
Next came the parameter values: $a \in (0;1)$. Reported charts:

```
{.8,.7,.6,.5}→L1
   {.8 .7 .6 .5}
```

```
WINDOW
 Xmin=█.5
 Xmax=5
 Xscl=1
 Ymin=-.5
 Ymax=5
 Yscl=1
 Xres=1
```

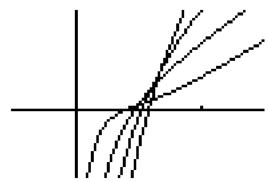


```
WINDOW
 Xmin=-.5
 Xmax=2
 Xscl=1
 Ymin=-.5
 Ymax=2
 Yscl=1
 Xres=1
```



```
{.4,.3,.2,.1}→L1
   {.4 .3 .2 .1}
```

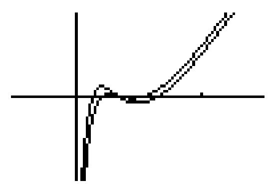
```
WINDOW
 Xmin=█.5
 Xmax=1.5
 Xscl=1
 Ymin=-.2
 Ymax=.3
 Yscl=1
 Xres=1
```

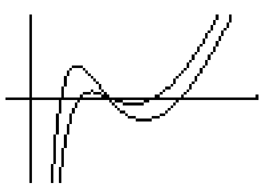


As Nathan argued that this range has only one solution I suggested to make a chart for: $a = 0.05$ or $a = 0.06$.

```
{.06,.05}→L1
        {.06 .05}
█
```

```
WINDOW
 Xmin=-.5
 Xmax=1.5
 Xscl=1
 Ymin=-.1
 Ymax=.1
 Yscl=1
 Xres=1
```





```
WINDOW
 Xmin=-.1
 Xmax=1
 Xscl=1
 Ymin=-.03
 Ymax=.03
 Yscl=1
 Xres=1
```

The resulting graph surprised the student. On the number axis for the parameter a noted the number of zeros:

Exercising many graphs on the calculator for the value close to $a = 0.06$ and analysing the number axis line for the parameter a, the student noticed that up to certain parameter values and the function has three numbers of zeros and from that only one. Nathan tried to determine the value of the parameter. For this purpose, performed on his calculator many graphs for different values of the parameter. The whole time he was focused on finding the value choosing the division range method. Examining these charts stated that it is difficult to determine the value. He marked it by A.

On the job card presented chards from the Calculator window:

He formulated a proposal.

Then I gave the student mysterious values: $A = e^{-e}$ i $B = e^{\frac{1}{e}}$. Student asked about the number of: e, boasted remaking the lesson forum: logarithms, read the book: Bogdan Miś, *Tajemnicza liczba e i inne sekrety matematyki*, WNT, 04, 2008.

Once again, we analyzed together the entire job giving the correct solution to the problem:

How many elements has the equation: $a^x = \log_a x$

$1 < a < e^{\frac{1}{e}} \quad a^x = \log_a x$ – equation has two solutions
$a > e^{\frac{1}{e}} \quad a^x = \log_a x$ – There is no solution
$a = e^{\frac{1}{e}} \quad a^x = \log_a x$ – There is one solution
$0 < a < e^{-e} \quad a^x = \log_a x$ – equation has three solutions
$e^{-e} \leq a < 1 \quad a^x = \log_a x$ – There is one solution

At the end Nathan made a function of the amount of solutions depending on the parameter a and set approximations of numbers: A and B.

3. Conclusions

The described experiment shows how the graphing calculator helped the student in solving the tasks of the problem, and served an important role in its termination at the level of knowledge and skills of the student. Shows how work with it develops skills and abilities to verify the hypotheses through a mathematical experiment. Reveals the ability to perform different experiments gave the converter, which observation and analysis of matched cases, allowed not only to explore certain regularities, formulate hypotheses about the problem being solved, but also gave the opportunity to verify these hypotheses, justifying the choice of the appropriate direction of research and finding a solution to the problem. After observing the student's involvement in the work can be seen as it increases the student's interest in the problem under consideration, so that the student understood it well and wanted to solve, produces an increase in excitement, satisfaction with work. Working with the calculator let the student find himself in the situation similar to the work of creative mathematics by giving satisfaction, self-confidence, increase self-esteem.

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PUPILS' MOTIVATION IN MATHEMATICS TEACHING USING THE CLIL METHOD

JITKA LAITOCHOVÁ, JAN WOSSALA

Abstract

The CLIL method is a relatively new trend in education. It is a combination of content and language learning (from the English: Content and Language Integrated Learning). The article is focused on using the CLIL method in the Czech Republic. The article explores the influence of this method on students´ motivation, too.

1. Introduction

One of the trends of the current labour market is migration in order to find a better job. However, this is associated with the need for high-quality language education not only in common, day-to-day areas but also in the specialized field in which one wants to work. One of the possibilities is to study abroad for a while to gain specialized knowledge in a foreign language. Another option (which may moreover serve as preparation for studies abroad) is the application of the CLIL method in teaching.

2. Using CLIL Method

The abbreviation CLIL stands for Content and Language Integrated Learning. Simply put, it means teaching a non-language subject in a different language than the native language. This method develops not only the ability of students to talk about common topics but also the method develops their ability to discuss subject-specific topics in a foreign language with subject-specific vocabulary.

The concept of CLIL was first officially used by David Marsh at a university in Finland in the year 1994. The official document which served to

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apply the CLIL method in the Czech environment was the National Programme of Teaching Foreign Languages in the Czech Republic for the Period Between 2005 and 2008 and it was prepared as a reaction to the Action Plan for the period between 2004 and 2006 issued by the European Commission. In 2009, the Ministry of Education, Youth and Sports issued a document entitled Content and Language Integrated Learning in the Czech Republic.

A project which aims to determine the use of the CLIL method at primary schools and high schools in the Olomouc Region and the South Moravian Region is in progress and here is part of current results.

First part of my pre-research was for directors of schools. Questionnaires were sent via e-mail to the mentioned schools. The objective was to find out whether the schools use the CLIL method, and if they do, in which subjects. The return rate was between 16% and 18% in both regions. Out of the total number of replies regarding use of the CLIL method in the Olomouc Region, 10% were positive. In most cases, the foreign language intended for integration into other subjects was English, only in one case it was French. Almost all subjects were represented that were taught or to be taught in the foreign language and no subject significantly prevailed over others. The situation was different in the South Moravian Region. There were 13% of replies positive. The English language was dominant and a wide scope of subjects were represented. But a major difference was in the dominance of some subjects – the most frequent were Mathematics, Arts, Music, Civics and Physical Education. Among other frequent subjects were Informatics, Social Sciences, Natural Science and History. These findings indicate that the CLIL method is not completely integrated and there is a frequent fear that such lessons are too difficult and the preparation for them too demanding.

In second part of my pre-research, I focused on the effect of the CLIL method on teaching mathematics. With regard to the fact that it was a pre-research, the sample consisted of 79 respondents – the pupils of the secondary level at a South-Moravian elementary school where the CLIL method had already been used for some time. Out of the 79 respondents, 32 were boys and 47 were girls. The pupils were given a questionnaire containing 22 statements with respect to which the pupils were supposed to express agreement or disagreement using a four-point scale. The questions focused on the climate in classes, the popularity of mathematics and whether it is worth it to use the CLIL method in classes. It was obvious that the pupils had good mutual relationships and there was a friendly atmosphere. The pupils also denied any derision in the event of failure of one of their classmates. This indicates a good climate for learning. The pupils also stated that the teachers were devoted to the subjects they taught and that the pupils were given tasks which they considered solvable. No classes thus indicated any problems, not even as far as mathematics teachers were concerned. The latest mid-year mathematics marks ranged from 1 to 4. This means that no pupil failed the subject. The general interest in mathematics was rather below average. However, when assessing mathematics taught by means of the CLIL method, there was a positive shift compared to lessons in which this method was not used. One of the frequent misgivings in connection with the integration of a foreign language into the teaching of non-language courses concerns the fear that pupils will not understand the task set in a foreign language. This was not confirmed in the case of these particular respondents, and hardly any of the pupils expressed any fear of non-understanding. What was also above average was the assessment of the pupils' activity; according to them, their activity in such lessons was increased. On the other hand, the lessons taught using the CLIL method were less focused on the actual subject in the pupils' opinion. This may be attributed to the fact that if foreign language activities take the form of games, pupils learn from such activities but do not see them as traditional lessons. This discrepancy between activity and focus on the actual subject deserves further research. Nevertheless, the assessment of the benefit of the method for the pupils' future lives was positive and slightly above average. In the use of a foreign language in mathematics, the pupils thus saw an advantage, whether in terms of their prospective studies abroad or their ability to discuss the issue of mathematics in the given foreign language. As regards the differences between the two sexes, they were not very profound. A relatively significant difference was found in the popularity of mathematics, where boys in general liked the subject considerably more than girls did. On the contrary, the popularity of mathematics taught using the CLIL method slightly decreased with boys, while girls liked the subject much more when taught in this way. Boys assessed their activity more positively than girls did, and participated more actively in the lessons. On the other hand, girls were generally the ones who found learning mathematics using the CLIL method beneficial.

3. Final remarks

Despite all its shortcomings, the CLIL method may bring positive aspects into teaching that above all consist in the preparation of students for studying or working abroad, where they will have to be able to discuss specialised topics, in addition to everyday conversation. The CLIL method might also make lessons more interesting, thus possibly increasing the attractiveness of non-language subjects.

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NON-STANDARD TASKS IN MATHEMATICAL EDUCATION

JOANNA MAJOR, MACIEJ MAJOR

Abstract

Mathematical knowledge and skills have been playing a more and more important role in our daily lives. At the same time, solving tasks is the essence of mathematics understood as a field of human activity. The subject of this paper are selected issues concerning some atypical tasks which play important role in mathematical education. Presented task is related to the elementary knowledge of probability theory.

1. Main consideration

Solving tasks states the backbone of teaching mathematics at every level of mathematical education. Having in mind the goal of preparing pupils and students, in the course of the educational process, to living in the surrounding reality, one should emphasize the tasks which allow the pursuit of general objectives of mathematical instruction, i.e. those that develop skills and attitudes necessary to a modern person, regardless of his or her field of activity. This goal may be achieved by assigning non-standard tasks to pupils and students.

This paper presents one non-standard task. The task is addressed to students of final grades of secondary schools that are preparing for the final secondary school examination (so called matura) at the advanced level, as well as undergraduate students majoring in mathematics. This task is composed of a problem and two different solutions. The students are asked to verify which of the given solutions is correct.

Solving this type of tasks develops, among others, intellectual attitudes evidenced by logical, creative, and independent thinking as well as by overcoming difficulties, and can improve the ability to analyse the content of

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the task and understanding of the global structure of the task. Moreover, as Polya notes, in mathematics itself, skills are more important than knowledge. What in mathematics does mean skill? It is the ability to solve problems, and not just typical tasks, but also those that require independent judgement, judgement ability, originality, creativity (see [5]).

In the light of the research conducted by the authors, assigning this type of tasks to students is a valuable part of mathematical instruction. The results of the research demonstrate that even able secondary-school and undergraduate students encounter considerable difficulties in solving non-standard tasks (see [3], [4]).

Let us consider the following task (see [1]) that describes a certain real-life situation and refers to the basics of the probability theory.

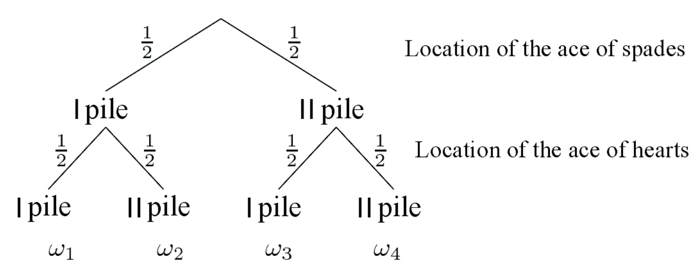
Task

A deck of 52 cards has been shuffled and divided into two piles of 26 cards each. What is the probability that the ace of hearts and the ace of spades are in the same pile?

Two solutions of the problem are presented below. Verify if they are correct and justify your answer.

Solution 1.

We identify two cards in the deck: the ace of hearts and the ace of spades. Now we analyse in which ways both cards may be distributed in two piles. The following stochastic tree will help us to conduct the analysis.



We have $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where:

ω_1 – the ace of spades and the ace of hearts are both located in pile I,

ω_2 – the ace of spades is in pile I and the ace of hearts is in pile II,

ω_3 – the ace of spades is in pile II and the ace of hearts is in pile I,

ω_4 – the ace of spades and the ace of hearts are both located in pile II.

In the model illustrated by the stochastic tree, the event A that both aces are in the same pile consists of two out of four outcomes with equal probability, symbolically $A = \{\omega_1, \omega_4\}$. Hence

$$P(A) = \frac{1}{2}.$$

Solution 2.

Note that dividing a deck of cards into two equal piles may be achieved by drawing 26 cards and considering the remaining 26 cards the other pile. Let us define Ω as the set of 26-element subsets of the set of 52 cards. Then the event

$A = \{$both aces are in the same pile$\}$

may be presented as $B \cup C$, where

$B = \{$both of the ace of hearts and the ace of spades have been drawn$\}$,

$C = \{$neither ace of hearts nor ace of spades has been drawn$\}$.

Events B and C are disjoint; hence

$$P(A) = P(B \cup C) = P(B) + P(C) = \frac{\binom{50}{24}}{\binom{52}{26}} + \frac{\binom{50}{26}}{\binom{52}{26}} \approx 0,45 \neq \frac{1}{2}.$$

As the presented solutions lead to two different outcomes, a conclusion can be drawn that at least one of them is incorrect.

The discussed problem is related to the construction of the mathematical or, more strictly, probabilistic model of the real-life situation. In essence, it must be verified if the probabilistic space assumed as the model of the experiment complies with this experiment, i.e. if it is an appropriate model of the given experiment.

Let us return to the experiment d described in the task, i.e. a random division of the deck of cards into two equal piles. The experiment's outcome consists of a pair of 26-member injective sequences $((a_n), (b_n))$, $n \in \{1, 2, \ldots, 26\}$ whose members belong to the set of 52 cards where $a_i \neq b_j$ whenever $i, j \in \{1, 2, \ldots, 26\}$ (no member of the first sequence is a member of the second sequence and vice versa). Note that every outcome, i.e. every pair of sequences may be identified with a certain permutation of the set of 52 cards, e.g. constituting the sequence $a_1, \ldots a_{26}, b_1, \ldots b_{26}$. Moreover, every permutation of the 52-card set may be interpreted as an outcome of the discussed experiment d (where the permutation is "split in two"). Consequently, there are altogether 52! outcomes of the experiment d; therefore, the probability of each of them is equal to $\frac{1}{52!}$. In the discussed model, the event $A = \{$both aces are in the same pile$\}$ contains the outcomes (pairs of sequences) where both defined elements of the 52-card set (the ace of spades and the ace of hearts) are members either of the sequence (a_n) or of the sequence (b_n).

As far as the discussed event is concerned, the order of putting cards in the piles does not matter. What is significant, is which cards are placed in the first and which in the second pile. Note that knowing the cards in pile one means implies knowing the cards in pile two. Hence, in order to determine the occurrence of event A, it is sufficient to draw 26 cards out of the deck (constituting the first pile, whereas the remaining cards will be in the second pile) and to control if the ace of spades and the ace of hearts are among the cards drawn. If both aces are in the first pile or neither of them has been drawn, the event A has occurred. The above discussion leads to the conclusion that the described experiment d may be substituted by another experiment, d_1, i.e. simultaneous drawing of 26 cards out of the 52-cad deck, that is the experiment described in Solution 2. Therefore, Solution 2 contains the correct result (see [2]).

As already mentioned, the discussed problem is a sample task of the type that should be assigned more often to those learning mathematics. The two contradictory solutions provoke the necessity of a deeper reflection on the problem and its solution. In particular, the task requires reflection on the construction of the described mathematical model and verification of its compliance with the situation in question. As it has been proved by our research, even college students majoring in mathematics are often not prepared for this kind of discussion.

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INQUIRY BASED MATHEMATICS EDUCATION (IBME) AND ITS REFLECTION BY PRIMARY SCHOOL TEACHERS

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Abstract

In the paper we present selected outcomes of a research into possible acceptance of inquiry based mathematics education as a method used by primary school teachers. We suggest some objective as well as subjective reasons for explaining its so far low level of usage and aim at identifying obstacles, which prevent the enquiry based techniques in the primary school educational environment.

1. Introduction and theoretical basis

In the discussion on possible system changes in teaching mathematics and (not only) mathematical education of primary school teachers there is among others one imperative: the need for quality of professional competencies of the teacher. It has been shown in everyday practice that the quality of mathematical teaching depends to a great extent on the didactic competence in the subject as the "core" of professional competencies of teachers, which makes teachers different from (and impossible to substitute by) other professionals. This includes especially the knowledge of the teaching content and various ways of its didactic processing *(Helus, 2001)*, knowledge of the curriculum and ability to apply this knowledge in teaching and also the competence to react and respond to students' performance in class in a qualified way, and the ability to makes use of this in teaching *(Tichá, 2012)*.

There are many challenges which suppose developed professional competence of the teacher and at the same time aim at their further development. Inquiry based education is one of these. However, as *Hošpesová (2014)*,

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points out, in the Czech environment, this concept is sometimes understood in a rather vague or even naive way. The words "inquiry based" often have (especially for the non-teaching public) incorrect connotations as they imply idea of passing latest findings of science to elementary school pupils, which – naturally – seems to be impossible in mathematics. Thus by inquiry based mathematics education we understand teaching inspired by research and inquiry techniques as a "purposive process of stating problems, critical experimenting, judging alternatives, planning, study and verification, making conclusions, searching for information, making models of studied events, discussion with others and forming coherent argumentation" *(Linn, Davis, Bell, 2004, p. 15)*, i.e. teaching containing activities focused on study and discovery. For more details on this method see *Stuchlíková (2010)*, *Dostál (2013)*, in Czech primary school education context *Hošpesová (2014)*, which both include relevant foreign sources.

In our paper we follow in the study done in papers *Fleková and Novák (2013)* and *Nováková (2013)* and include several findings from a questionnaire research into opinions of primary school teachers on possibilities of using inquiry based education at primary schools.

2. An example of inquiry based education in the elementary school context

In order for the respondents to recall the concept of the inquiry based education (IBE or IBME further on), we included the following example in the questionnaire: working out the number of vertices (v), sides (s) and edges (h) using models of solids which was meant to result in finding out the Euler's theorem $v + s = h + 2$ (for details see *Fleková, Novák 2013)*.

This tasks assumes the common knowledge of pupils at the end of their elementary education – basic solids (convex polyhedrons), naming the polyhedrons, concepts of vertex, side and edge.

Stages of discovery:

(1) *Setting the task for students:* Create models of tetrahedron, cube, regular square pyramid, regular five-sided pyramid, and hexagonal prism. Look for the relation between number of sides edges and vertices in each polyhedron.
(2) *Experiment realization and its recording (pupils do themselves based on teacher instructions):* Write the number of vertices (v), sides (s) and edges (h) of each polyhedron in a table and try to find out the relation between v, s and h.
(3) *Inquiry – discovery (with teacher's help):* From the data in the table we have found out that

- The greatest number for all solids is the number of edges. Furthermore, it is obvious that
- If we make the sum of two smaller numbers on each row (in all cases $v + s$), we get a number which is only slightly different from h, the number of edges. How slightly? Is the difference the same for each row?
- After a closer look we find out that there holds $v + s = h + 2$.

(4) *Hypothesis formulated – by pupils or with teacher's help:* In an arbitrary convex polyhedron with v vertices, s sides and h edges there holds $v + s = h + 2$ or $v - h + s = 2$.

(5) *Hypothesis verification:* Make models of some other solids such as cuboid or octahedron and verify the hypothesis: does the above assumption hold for these solids as well?

number of	tetrahedron	cube	regular square pyramid	regular five-sided pyramid	hexagonal prism
vertices (v)	4	8	5	6	12
sides (s)	4	6	5	6	8
$v + s$	8	14	10	12	20
edges (h)	6	12	8	10	18

Table 1: *Number of vertices, sides and edges of convex polyhedrons*

3. Objective and method

We wanted to find out whether teachers know the idea of inquiry based education and whether they (based on their own pedagogical experience) regard it as suitable for teaching mathematics at elementary school. We also wanted to find out topics which have the greatest potential for using IBME (and in which year can these be taught) and what obstacles prevent IBME from being used.

Our research was conducted in a group of 72 respondents, which included both experienced teachers of faculty primary schools at which students of the attended form of study do their teaching practice (16) and students of 3^{rd}–5^{th} year of the combined form of study who were mostly teachers actively teaching at schools yet who at the same time were receiving or expanding their primary school teaching qualification during their lifelong learning (56).

The following two research tools were used:

(1) Our own questionnaire which consisted of 20 closed-ended or semi-closed-ended questions: 8 questions focused on personal characteristics of respondents, further questions aimed at establishing respondent's knowledge of IBE, their to-date experience with this teaching method and obstacles which prevent teachers from its wider application in teaching. As for advantages of questionnaire research are concerned, it has been often suggested that *(Chráska, 2007)* it enables the carrier to gather great amount of data in a short time period. On the other hand, the possibility of including personal opinions of the respondents is limited. In order to minimize this handicap, we included the semi-closed-ended questions. We also supplemented the research with

(2) Interviews of selected respondents (students of the 4^{th} or 5^{th} year of study).

We are fully aware that the relevance of our findings is limited due to our choice of the method and sample of respondents. Yet even in spite of these drawbacks we believe that it gives useful feedback and is a good tool which can be considered in the professional training of teachers.

4. Selected findings

From the answers there follows that vast majority of the teachers is not familiar with the concept of IBE yet would like to learn it. This is the answer given by 84% of the respondents. At the same time the knowledge of IBE is considerably higher for teachers of the faculty schools (56%). This can be justified by the fact that such teachers usually learned about the method during the optional courses they had taken.

The method was mentioned as suitable for primary mathematical education by 87% of teachers, more than half (53%) would use it when working with talented pupils. However, the answers might be influenced by equalling the method with the demonstrative example which was included in the questionnaire. Respondents also gave their reasons for implementing IBE. Most often: it increases motivation of pupils to learn mathematics, makes pupils be interested in the subject matter, may help in better understanding of the subject matter, pupils "absorb" knowledge better, pupils' knowledge is more durable.

Among the main reasons against IBE there were: the fact that it is time-consuming both in class and in teacher preparations (67%), the fact that prior to using it suitable conditions and tools must be created (47%). Respondents also are not convinced that the IBE teaching is more efficient and gives better results than the usual frontal teaching.

We were also interested in how often would respondents include IBE in their teaching if they had this opportunity. Vast majority of respondents (91%) would include it scarcely, only in some special classes when teaching certain suitable topics.

Topics suggested by teachers for the potential use of IBE in teaching include most often the triangle inequality, the correspondence between area and circumference of planar geometric objects such as square; puzzles, numerical quizzes, etc.

The decision to include IBE in teaching depends mainly on the possibility to use sufficient amount of specific topics and ideas (the answer given by 87% respondents). The answer "depends on my own mathematical knowledge and didactic competencies" was chosen by 9% of respondents only.

The follow-up interviews of selected respondents clearly indicate interest in the inquiry based education which is seen as important motivational tool especially when working with talented pupils. However, the respondents warn that application of the method assumes developed pupils' competence to solve problems as well as competence to work such as to perform measurements, search for data and interpret data. This to some extent means that the method is not suitable for the very young pupils.

Topics suitable for IBE were discussed in the interviews as well. Only at the intuitive level, without previous verification, the following topics were suggested: properties of arithmetical operations (commutativity, associativity, existence of the neutral element), relations or rather regularities in number or image series, sorting geometrical shapes according to their properties. As one of the teachers frankly pointed out, "anything can be used, it all depends on the teacher".

5. Final remarks

We do not hold findings of our research for surprising. Yet even though it would be incorrect to state firm conclusions, we believe that certain points may be deduced:

- Inquiry based education as one of current constructivist directions is neither sufficiently known nor used in the primary school education environment. However, when acquainted with it (usually in some optional courses in the lifelong learning), teachers welcome it and try using it in their teaching. Our findings correlate with findings from Slovakia *(Pavlovičová, 2014)*.
- Obviously, the reasons limiting or preventing the use of IBE are seen as "outer", "objective" ones by the teachers; this mostly includes great time-consumption preventing "teaching the subject matter", insufficient amount of didactic materials, etc. while low level of

didactic competence or didactic quality of the teaching content are marginal.

- We were alarmed to have found out that the respondents had no chance to learn about IBE in detail. If one agrees that IBE should be included in teacher training, one must look for suitable ways of implementing it in teaching. One of the attitudes, which is based on students' own experience with inquiry based activities and their reflection has been mentioned by *Hošpesová (2014)*.
- In our research we have not reflected one of the important factors of potential application of IBE – what the attitudes of pupils are? We believe that on the pupils' side there sometimes exists insufficient motivation or problematic background of necessary mathematical knowledge and skills. Yet also here the role of the teacher is the key one. As *Dostál (2013)* points out, pupils who cannot proceed when solving problems and as a result are at a loss after a few unsuccessful attempts, must be helped by the teacher by questions, assistance or advice so that they are able to construct their own authentic image of the world, built on their own experience.

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MANY FACES OF MATHEMATICAL MODELLING

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Abstract

Mathematical modelling is a concept that covers a wide range of activities. Mathematical modelling can be understood both as formulation of an equation, a function, etc., describing a given situation and as a whole process of creating a model, starting from the real-world situation to the creation of a ready-to-use optimized tool. The work presents different approaches to mathematical modelling from the point of view of teaching mathematics. It presents the results of the research conducted on students (future teachers) regarding their theoretical knowledge and skills related to mathematical modelling.

1. Introduction

While studying the problem of mathematical modelling, I discovered that the concept itself is differently perceived by its users: biologists, statisticians, athletes, economists. I am interested in differences in understanding of this concept from the point of view of teaching mathematics. Depending on the level of teaching various activities can be understood as mathematical modelling. For a teacher modelling can be, for example, writing down an algebraic equation or function formulas which illustrate a given situation. On the other hand, for students (e.g. of economics), it will be a complicated process of formulating the influence of various parameters on the value of the considered variable using mathematical tools. Both interpretations comply with the definition of modelling because it covers a wide spectrum of activities which can be differently understood at different levels of education. How then define the considered notion for the purposes of teaching mathematics?

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2. The definition of mathematical modelling

W. Blum and R. Ferri define mathematical modelling as "the process of translating between the real world and mathematics in both directions" (2009, p. 45).

Z. Krygowska, who laid the foundations of Polish didactics of mathematics, defines mathematical modelling in the following way: "Mathematical modelling is an ability to describe a real situation in the language of mathematics, to interpret and verify the results in natural language, to match ready-to-use mathematical models with real situations and to search for real situations that are specific to those models, to reflect upon, analyse and evaluate one's own and the others' mathematical models. The construction of a mathematical model of the situation requires the student to examine it, and then to distinguish the objects and relations between them."

M. Niss writes concisely: "When a mathematical model is introduced (selected, modified or constructed) from scratch to deal with aspects of an extra-mathematical context and situation, we say that mathematical modelling is taking place". (2012, p. 50).

The above-mentioned statements show a wide spectrum of activities related to modelling. This term refers both to the selection of a suitable tool of solving the problem as well as the entire process of creating a new model. The quoted statements indicate the need to verify the chosen mathematical tool, control various aspects that are included in the tool and the model, verify the correctness, even by choosing sample data or considering various possible solutions to the problem, etc. Mathematical modelling is thus a process in which one cyclically returns to the initial real situation. This process is to realize which aspects of the reality have to be included in the model and which should be simplified.

Now we can see that mathematical modelling is an activity that consists of many elements. Depending on which element of this process will be given greater attention, mathematical modelling can be differently perceived.

3. Mathematical modelling in school education

W. Blum writes that "In Germany mathematical modelling is one of six compulsory competencies in the New National 'Educational Standards' for mathematics. However, in everyday mathematics teaching in most countries there is still only few modelling. Mostly 'word problems' are treated where, after 'undressing' the context, the essential aim is exercising mathematics." (Blum, Ferri, 2009, p. 47).

In Poland, mathematical modelling is one of several groups of mathematical learning objectives included in the core curriculum. In accordance with this act, students should possess the ability of mathematical modelling as they finish the fourth stage of education (age of 19). For students on mathematical education at the advanced level, this requirement is formulated as follows: a student creates the mathematical model of a given situation, taking into account limitations and reservations (Core curriculum with commentary, 2009, p. 41). However, in Polish schools, mathematical modelling is generally associated with the ability to use mathematical tools (e.g. systems of equations, linear functions) to solve the so-called word problems.

Here are some examples of tasks related to mathematical modelling indicated by the authors of the page
http://www.e-zadania.pl/matura-2012/bardzo-wazne-materialy
/standardy-wymagan-egzaminacyjnych/modelowanie-matematyczne/
which aims at preparing secondary-school graduates for their exam (matura):

1. Show number 42 as a sum of two components so that the difference of their squares equals 168.
2. Point $B = (-1, 9)$ belongs to the circle tangent to the axis Ox at the point $A = (2, 0)$. Find the equation of this circle.

In my opinion, such understanding of mathematical modelling is too narrow: tasks presented here are closed and explicitly relate to very specific mathematical issues (e.g in task 2 one has to relate to the equation of a circle known from analytic geometry). Also exercises in mathematics textbooks are generally closed due to the data; they do not require selection of information and often explicitly indicate a mathematical tool. Unfortunately, mathematical modelling is often perceived like this by teachers, which results from the fact that such understanding is imposed on them by the sources available to them.

In educational journals other faces of mathematical modelling can be found because examples of real situation appear in them. We can distinguish here two types of work:

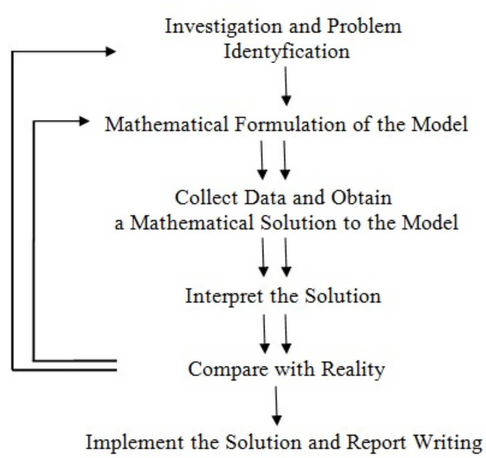
- Searching for a model based on the specific data
- Creating a model from scratch, knowing only a problem from reality

Creating a mathematical model of a phenomenon (problem) on the basis of specific data comes down to the search for functional dependencies between the data. Found model (function, equation, system of equations) allows us to predict the course of events in such moments in the future for which we may not have empirical data. The most commonly used models are functional models in the form of linear, polynomial (square in particular) and exponential functions. The search for a model can be done e.g. by a computer program which has the option of retrieving functions representing specific data. Therefore, mathematical modelling can be seen as a computer simulation performed by an appropriate program to get the formula of a function describing the considered phenomenon (see: Rybak).

Already at the stage of schooling, mathematical modelling can be understood as a search for a model from scratch, starting from the analysis of the problem, collecting sample data on which the model is created by testing and verifying the design model. Examples of school problems solved in such a way can be found in the work of P. Zarzycki (2009). Despite numerous educational benefits of unassisted creating of a mathematical model, such approach is absent from school textbooks, not to mention even the attempts to create the general model by searching for and making a symbolic notation of relations within the considered problem.

4. Mathematical modelling in university teaching

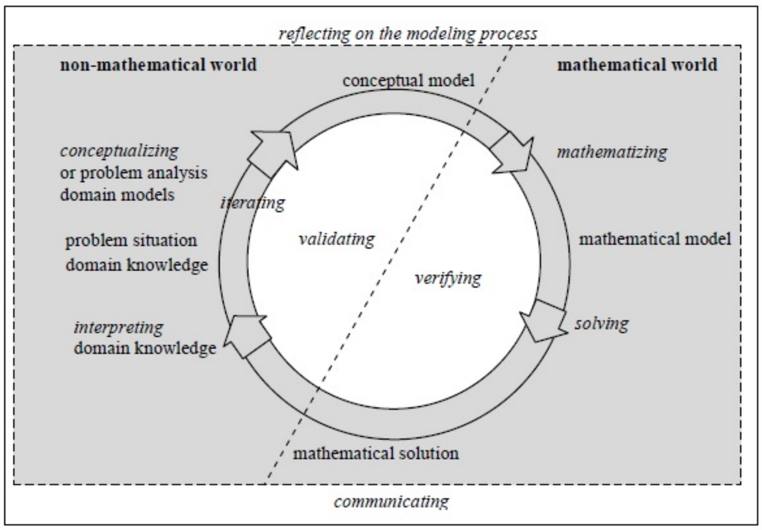
At the university level, students can meet with mathematical modelling – this applies to certain fields of study. In Poland, they are mostly technical and economic fields. Unfortunately, modelling hardly ever appears in Polish mathematical studies, even in those with teaching specializations.



Picture 1. Stages in the process of modelling (W. Morris 1967)

In the scientific literature, one can find much information both about learning and teaching mathematical modelling (Blum, Ferri, Niss, Warwick). Scientists through mathematical modelling understand the whole process of creating a model from choosing a problem (situation) to producing the final specification of a ready, verified and properly working model. In the literature one can find many diagrams showing the stages in the process of mathematical modelling. The differences in these patterns arise mostly from different levels of detail, but they always take the form of a cycle and regardless of the amount of steps shown, they describe the same process that takes into account both mathematical and non-mathematical (realistic) aspects of a given situation.

In 1967, W. Morris illustrated the process of modelling in the form of two loops consisting of 6 stages (see picture 1). The loop structure of the modelling process causes that certain stages of the process are repeated in order to create the best model.



Picture 2. The process of mathematical modelling (J. Perrent, B. Zwaneveld 2012)

In 2012, J. Perrent and B. Zwaneveld presented their diagram of mathematical modelling, which was created after a thorough analysis of a number of diagrams showing elements of mathematical modelling (see picture 2). Their draft is consistent with Morris's proposal although it was limited to a single loop. The new 'modelling cycle' clearly indicates the interpenetration of the stages (actions, activities) leading to the creation of a mathematical model. Particularly evident is the suspension of the process in two areas: in the world of mathematics and the real world. Activities performed in the first of them are known from math classes, while moving in the other requires a higher degree of skills. It results from the fact that the creator of a mathematical model should carefully analyse the problem so as to see what elements of the reality influence the analysed variable and know what dependencies there are between the analysed variables.

After mathematization and performing the calculations, a verification and validation of the results is needed. The authors of the diagram explain these activities in the following way (Perrenet, Zwaneveld, 2012, p. 18):

- Verification: The mathematical model and the solution have to be tested and adapted against mathematical logic and consistency.
- Validation: The mathematical model and the solution have to be tested and adapted against the requirements of practice.

It is these activities (among other things) that Niss (2012) thought about when he wrote that mathematical knowledge alone is insufficient for efficient creation of a mathematical model. The representations of mathematical modelling shown above cover the entire process of modelling, which should be carried out independently and not with the help of a computer program. They indicated these aspects of the process of modelling which require skills that go beyond just a deep understanding of mathematics, i.e. understanding of a real problem, creating a model that is based on reality, creating a mathematical model from a real model, interpreting mathematical results with reference to a real situation. So we can see that scientists, despite using different forms of showing what mathematical modelling is, unanimously refer to the same range of activities.

5. Mathematical modelling in the consciousness of the students of mathematics

In mathematical studies in Poland the subject of "mathematical modelling" rarely appears. This also applies to fields with teaching specialization, which is quite surprising given to the fact that as future teachers, students will be required to teach skills related to modelling, all the more so because the competence in mathematical modelling is different from pure mathematical skills, which Niss (2012) and Maaß(2006) prove in their works. So how do future teachers understand mathematical modelling if they are not quite familiar with this concept? I tried to answer this question by conducting a research.

5.1. **Methodology of the research.**

5.1.1. *The objectives of the research and the description of the research tool.* The aim of the research was to examine the knowledge and skills of students of mathematics (future teachers) related to mathematical modelling. The students were asked to solve two tasks and fill out the questionnaire. The initial 4 questions focused on the way of solving the tasks, whereas the last four focused on the students' knowledge in the field of mathematical modelling. These two types of questions were designed to examine both skills and theoretical knowledge in the field of mathematical modelling.

The research tool consisted of two cards – the first of them with the tasks to be solved and the second one with the questionnaire with questions and empty space for answers. The content of the tasks was the following:
Exc. 1 Kasia wants to prepare yoghurt sauce. For this, she went shopping. Unfortunately, on her way back home the storm started. At home, it turned out that the label on the purchased yoghurt is partially soggy and thus partially illegible. Is Kasia able to know the capacity of the yoghurt container if she read from the label the following information:

Nutritional values:	100 gr.	Container
Energy (cal.)	75	113
Fat (gr.)	1.5	2.25
Calcium (mg)	160	240

Exc. 2 Some kind of curd contains 5% of fat. Complete the following table by calculating the amount of fat in individual packs.

Pack (gr.)	Fat (gr.)
100	
150	
200	

The questions in the survey regarding the above tasks were as follows:
1. How did you solve the first task? What competence did you need to have to solve it?
2. How did you solve the second task? What competence did you need to have to solve it?
3. Which of the two tasks was more difficult? What was difficult?
4. How do the processes of solving each task differ? In what way are they similar?

The analysis of the written work of students was to find the answer to the following research questions:

1. Do the students see in these tasks the elements of widely understood mathematical modelling?
   - Do the respondents notice the difference between the types of tasks?
   - Are the respondents able to explain the difference between the approaches to the discussed tasks?
   - Are they able to name the skills needed to solve them?
   - Can they notice non-mathematical (non-computational) competence needed in exc.1?
2. How does the process of creating the model by the students look like?

- Is the cyclical nature of this process visible?
- Do the students check the results?
- Do they see the necessity of the analysis of the information in exc.1?

I expected that the students would see the difference in the didactic approach to solve the tasks. I hoped they would see that the second task needed only the application of the indicated mathematical tool, whereas the first one involved creating or selecting a mathematical model of the situation shown. I also expected that the respondents, while describing the skills used, would associate them with the competence related to mathematical modelling.

In order to recognize the students' knowledge on mathematical modelling, I asked the following specific questions:

- Are the future teachers familiar with the term "mathematical modelling"?
- What do the students understand by mathematical modelling?
- Do they see mathematical modelling in school education?
- Are the students aware of when mathematical modelling appears in school education?
- Do the future teachers know that mathematical modelling belongs to one of the main groups of teaching mathematics objectives?
- Are the students able to list the forms that a model can take?
- Are the respondents aware that modelling is a process?
- Do they know the stages of the modelling process?

I searched for the answers to these questions by analysing the students' statements from the questionnaire:

5. *How do you understand "mathematical modelling"?*
6. *In what form can a mathematical model be presented?*
7. *Where in school education does mathematical modelling appear?*
8. *What stages of the modelling process can you name?*

I was also interested in whether the students could see the relation between the questions from both parts of the questionnaire or whether questions related to modelling would suggest some relation and influence the answers for the questions from the first part of the questionnaire.

5.1.2. *Organization of research.* The research was conducted in January 2014 as part of classes on didactics of mathematics of the fourth stage of education. The questionnaire was completed by 12 second-year maths students of complementary master's studies with specialization in teaching mathematics and computer science. Time to complete the tasks was adjusted to the needs of the students and took about 30 minutes.

5.2. **The analysis of the respondents' work.**

5.2.1. *The analysis of the approach to solving the tasks and answering the questions from the questionnaire (1-4).* Two tasks constituting a part of the research tool were not difficult from the factual point of view – all the respondents did them correctly. It turned out, however, that they are significantly difficult from the didactic perspective. Although the research was carried out on future teachers, none of them recognized that the first of the two tasks requires an unassisted discovery of the research tool, which is understood as mathematical modelling. The students do not associate this task with mathematical modelling that requires: the analysis of the situation and the information given, selection of the data and evaluation of their relevancy, thinking over the way to solve it, bringing a real situation to the level of mathematical calculations by creating an appropriate model (proportions or equations in this case) and verification of the results. They focus only on computational solution and not on didactic aspects of solving the task. The respondents cannot show the difference between these tasks. None of them noticed that in the first task one has to choose the method of solving the task on their own, whereas in the second task it was explicitly stated that one has to use percentage calculations. Only one person noticed that the second task is purely a mathematical computational task in which non-mathematical content is irrelevant. He or she writes that the difference may lie in the fact that the first task is a typical instructional task that is based on the correct understanding, whereas the second one focuses only on calculations. It is not true, though, that the first task is a typical instructional task. Such perception of the task indicates traditional computational techniques (verification of the data, the unknowns, calculations), which may suggest that the students associate solving tasks more with performing calculations than with creating a model. Unfortunately, focusing only on the methods of calculation does not result in the ability to smooth mathematical modelling (see: Niss). However, the author of the above statement emphasizes the fact that in the second task the whole activity is reduced only to the correct application of a known and suggested calculation. This statement indicates that despite the lack of clear differentiation between the types of the tasks, students intuitively perceive the difference in the approaches to them. The statements of 10 persons suggest different attitudes towards considered problems. Although they cannot explain it, 8 persons say that the first task was more difficult, whereas 3 persons write that the second task can be solved in a mental calculation. The intuition of the respondents is visible in the amount of description – they pay more attention to task 1, they provide more sophisticated responses in the questionnaire or write down calculations and proportions, whereas in task 2 there are often only the results. The intuitive differentiation between the types of tasks is suggested in such statements as: "The first task was more difficult because one had to think longer about the instruction and carefully analyse the table so as to create a well-arranged proportion". "In the first task, one had to think more about how to arrange proportions well, whereas in the second task one had only to calculate the percentage of a given number". The underlined words suggest that the respondents felt that task 1 was more demanding – it was more time-consuming and involved more analysis of the data and thinking about the problem in it, creating the tool was more difficult, the correctness of this tool is not obvious, whereas in task 2 one had "only to calculate" – one did not have to think but to do the calculations mechanically. The statements of 7 persons suggest that they see in task 1 the necessity of the analysis of the information given – as required competence here they indicate the ability to read with understanding and logical thinking. In none of the works checking the results appears, but in two of them various elements were used to solve the task (fat, calcium, energy), which may indicate the attempt to verify the results. As a way of solving the tasks, the students indicate the use of proportion and the calculation of the percentage of a given number. They consider the knowledge of the above methods, as well as reading with understanding, logical thinking, knowledge of multiplication tables or the ability to convert units, as necessary competence, however none of them noticed in task 1 non-mathematical abilities, which Maaß, Perrenet and Zwaneveld point to.

The conducted analysis shows that the students, although they intuitively see the difference in the approach to the tasks, cannot formulate this and focus on calculations, not on teaching benefits that can be obtained from these tasks.

5.2.2. *The analysis of the part of the questionnaire related to the knowledge about mathematical modelling.* Future teachers should be familiar with the concept of mathematical modelling to teach their future students abilities related to it. The analysis of the students' written responses shows that their knowledge about modelling is little. I suppose this is because teachers do not attach importance to the very concept of modelling in the course of teaching. The skills related to it (according to teaching mathematics objectives in the core curriculum) should be taught at every stage of education and so it happens, however important information is usually overlooked that these activities aim at developing skills of mathematical modelling. Consequently, there appears lack of understanding of the concept of modelling, which can be confirmed in the students statements from the questionnaire. The analysis of the works showed that among 12 respondents only 5 of them show understanding of mathematical modelling that is close to its real meaning, of which 2 of these definition can be considered correct, e.g. "By mathematical modelling I understand a process of creating a pattern (searching for arithmetical dependencies) between certain variables (parameters) so as to simplify future calculations by application of data". The majority of the students (7 persons) associate mathematical modelling with creating a pattern, a formula, an equation or a function, which is a good association, but they are not able to precisely define what this term means. Unfortunately, there are also answers completely far from the truth, e.g. "I understand mathematical modelling as the way in which the teacher guides the student to the correct way to solve the task". In the interview conducted after completing the questionnaire the students admitted that they do not exactly know what mathematical modelling is but for 2 people who came across this concept on different studies.

Despite problems with explaining what mathematical modelling is, the majority of the respondents (9 persons) can see it in school education, however it is difficult for them to state where exactly it appears. Very general answers appear – on computer science, mathematical and natural subjects, during studies and in high school. Three of these answers are wrong and it results from wrong understanding of modelling, e.g. "on every maths class" – said one of the students who perceives modelling as kind of the teacher's gun guiding a student to the correct solution. Some generally formulated answers also appear which can be considered correct, e.g. functions, planimetry, stereometry. In none of the students' statements reference to the objectives of teaching mathematics appears, although mathematical modelling belongs to one of the groups of these objectives.

A mathematical model can take various forms and the students' answers prove this. Although only 6 persons answered the question about the form of a model, the majority of them were correct. Most frequently mentioned form was that of the graph and the equation with unknowns, but also such forms were mentioned as the description, diagram, inequality, block diagram. Among the incorrect answers were the example and the word problem.

Three persons emphasized that mathematical modelling is a process but none of them explained exactly why – students treat modelling as a single activity, without the necessity to test or improve the model. Four students made an attempt to list the stages of the modelling process. However, these concern only a single creation of the model and not the process of improvement of the working model. The mentioned stages are the following: choosing the problem, analysing the task, collecting the data, searching for and writing dependencies between the variables, solving equations. The student who mentioned the most stages was the one to mention the most of them and he or she described the process of creating a model the most precisely, however with limiting it down to creating only the first version of the model.

The conducted research reveals students' lack of appropriate knowledge in the field of mathematical modelling although they managed to do the tasks, one of which required the use of mathematical modelling. Only in two out of twelve works we can see that the respondent, while answering the questions related to modelling, tries to relate to the tasks done at the beginning. In the rest of works no dependencies between these two parts of the questionnaire appear.

6. Summary

Mathematical modelling is a concept that covers a wide range of activities. It is differently perceived by various groups of users depending on which activity they focus on. Mathematical modelling can be understood both as formulation of an equation, a function, etc., describing a given situation and as a cyclical process of creating a model that balances between the real world and the mathematical world.

Future teachers of mathematics have little knowledge on this subject. They cannot name the differences between a task that requires the skills related to mathematical modelling and a typical computational task, however their answers suggest that they intuitively see the difference in the approach to these tasks. Therefore, it appears that we should not delude ourselves that students will gain non-mathematical competence related to mathematical modelling but we have to explicitly show them the difference in the approach towards non-computational tasks which require the analysis of the real situation and where a decisive factor concerning, inter alia, the selection of information is important. One fact is comforting (as confirmed by researchers in the field of mathematical modelling) that one can effectively teach these skills that lie beyond pure mathematics, which can result in numerous didactic benefits.

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THE EVALUATION OF MATHEMATICAL TEXTBOOKS ACCORDING TO THEIR TEACHING QUALITY

MARKÉTA ŠNORIKOVÁ

Abstract

The author introduces one possible option for the analysis of the textbooks.

It introduces an analysis of those teaching methods used in four textbooks. The presentation is focused on the evaluation of two parallel mathematical textbooks for secondary schools of four different publishers by two publishers Didaktis and Prometheus. It evaluates and compares the textbooks. The aim of this evaluation is to help teachers with the selection of the most suitable textbook for teaching purposes.

1. Introduction

The textbook has an irreplaceable role in the educational process. Content and structural properties of the textbooks are changed. These changes are caused by new technologies and by the change of educational system as well, that includes the framework educational programmes.

Creating a textbook authors have to fulfil specific requirements in comparison with other texts. These requirements are imposed not only on the content of textbooks but also on the structure of the entire textbook, lucidity and its visual appearance. The textbook is considered a specific construct, which differs by its characteristic structural components. These structural components ensure the use of the textbook. [1]

Jan Průcha compiled a universal taxonomy of structural components identified in textbooks of any subject. The following diagram (Diagram 1) represents all structural components according to the taxonomy by Jan Průcha. [2]

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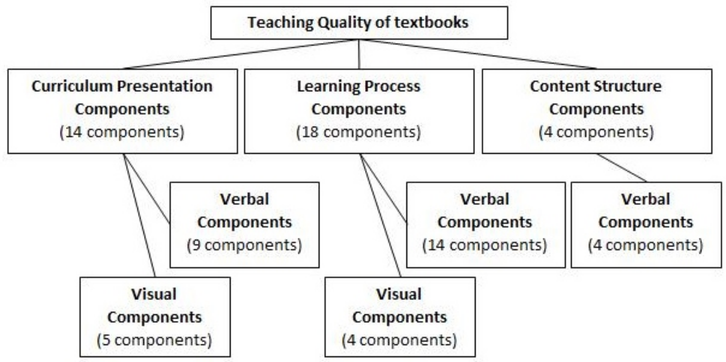

Diagram 1

Calculation of the textbook teaching quality is based on the occurrence of the structural components within a textbook. The textbook teaching quality decides the procedural efficiency of the textbook and it is expressed by quantitative factors:

1) **Partial Quotients of Teaching Quality**:
   - The Use of Curriculum Presentation Components (E I),
   - The Use of Learning Process Components (E II),
   - The Use of Content Structure Components (E III),
   - The Use of Verbal Components (E v),
   - The Use of Visual Components (E o),
2) **Total Quotient of Teaching Quality (E)**. [2]

The quotient is calculated as a percentage of the total number of components to the number of all possible components. It is expressed as percentage from 0 to 100%. The closer the value of educational facilities is to the upper limit, the higher the teaching quality is and thus greater procedural efficiency. The study of teaching quality of a textbook has two main objectives. Firstly, it determines the occurrence of structural components, and secondly, it corrects editing textbooks to enhance teaching facilities [3].

For rating purposes I have chosen four available mathematics textbooks:

- Krupka P., Polický Z., Škaroupková B., Květoňová M., Cizlerová M., *Matematika pro střední školy*, Základní poznatky, Učebnice. Vyd. 1. Brno: Didaktis, 2012, ISBN 978 – 80 – 7358 – 198 – 5.
- Odvárko O., *Matematika pro střední odborné školy*, Základní poznatky, Vyd. 1. Praha: Prometheus, 2009, ISBN 987 – 80 – 7196 – 394 – 3.

- Calda E. *Matematika pro netechnické obory SOŠ a SOU*, 1. díl. Vyd. 1. Praha, Prometheus, 1996, ISBN 80 – 7196 – 020 – 9.
- Bušek I., Boček L., Calda E., *Matematika pro gymnázia*, Základní poznatky. Vyd. 2, Praha, Prometheus, 199, ISBN 80−85849−34−8.

The textbooks were chosen intentionally. They are meant for secondary schools with the same thematic focus. They introduce the curriculum of the basic knowledge and were published in different years. The textbooks by Krupka et al. and Odvárko are newer and were released after the introduction of the framework educational programme. On the contrary, the textbooks by Calda and Bušek et al. were published prior to the introduction of the framework educational programme.

2. **Main results**

To survey the teaching quality we can use a form that contains all the structural components according to the taxonomy by Jan Průcha. The following table clearly shows which structural components the chosen textbooks contain. All textbooks are presented in Tab. 1, Tab. 2 and Tab. 3 for better comparison. For better differentiation we provide the year of the publication.

FORM OF TEACHING QUALITY	Did. 2012	Prometh. 2009	Prometh. 1996	Prometh. 1995
I. CURRICULUM PRESENTATION COMPONENTS				
A) VERBAL COMPONENTS				
1) Simple explanatory text	✓	✓	✓	✓
2) Clear explanatory text	×	×	✓	×
3) Year content summary	×	×	×	×
4) Theme content summary	✓	✓	×	×
5) Previous year content summary	×	×	×	×
6) Additional texts	✓	✓	×	×
7) Notes and explanations	✓	×	✓	✓
8) Subtexts to illustrations	✓	✓	×	×
9) Glossary of Terminology and Vocabulary	×	×	✓	✓
B) VISUAL COMPONENTS				
1) Art illustration	×	✓	×	×
2) Learning illustration	✓	✓	✓	✓
3) Photos	✓	✓	×	×
4) Maps, plans, graphs, diagrams etc.	✓	✓	✓	✓
5) Colour illustration	✓	✓	×	×
TOTAL	9	9	6	5

Tab.1

II. LEARNING PROCESS COMPONENTS	Did. 2012	Prometh. 2009	Prometh. 1996	Prometh. 1995
C) VERBAL COMPONENTS				
1) Foreword	✓	✓	✓	✓
2) Instructions how to work with textbook	✓	×	×	×
3) General stimulation	×	×	×	×
4) Detailed stimulation	✓	✓	×	×
5) Differentiation of the level of the curriculum	✓	✓	×	✓
6) Questions and tasks after a topic, a lesson	✓	✓	✓	✓
7) Questions and tasks after a study year	×	×	×	×
8) Questions and tasks before a study year	×	×	×	×
9) Instructions for tasks	×	×	×	×
10)Suggestions for extra curricular activities	✓	✓	✓	✓
11)Explicit expression of pupils' learning aims	×	×	×	×
12)Means for pupils' self-assessment	✓	×	×	×
13)Results of questions and tasks	×	✓	✓	✓
14)Links to other sources of information	×	✓	×	×
D) VISUAL COMPONENTS				
1) Graphic symbols	✓	✓	✓	✓
2) Using special colours for text	✓	✓	×	×
3) Using special type of font for parts of text	✓	✓	✓	✓
4) Using envelopes for schemes and tables	×	×	×	×
TOTAL	10	10	6	7

Tab.2

III. CONTENT STRUCTURE COMPONENTS	Did. 2012	Prometh. 2009	Prometh. 1996	Prometh. 1995
E) VERBAL COMPONENTS				
1) Content of textbooks	✓	✓	✓	✓
2) Division into thematic blocks	✓	✓	✓	✓
3) Marginals	×	×	×	×
4) Index (subject, name)	✓	✓	✓	✓
TOTAL	3	3	3	3

Tab.3

The following table (Tab.4) summarizes the total number of structural components that are assembled into groups to match the partial coefficients of textbook teaching quality.

TOTAL NUMBER OF STRUCTURAL COMPONENTS	Did. 2012	Prom. 2009	Prom. 1996	Prom. 1995
CURRICULAR PRESENTATION COMPONENTS	9	9	6	5
LEARNING PROCESS COMPONENTS	10	10	6	7
CONTENT STRUCTURE COMPONENTS	3	3	3	3
VISUAL COMPONENTS	7	8	4	4
VERBAL COMPONENTS	15	14	11	11
TOTAL NUMBER	22	22	15	15

Tab.4

From the observed number of structural components we can measure the textbook teaching quality. Proportional representation of partial and total coefficients are shown in the following table (Tab.5).

QUOTIENT OF TEACHING QUALITY (%)	Did. 2012	Prom. 2009	Prom. 1996	Prom. 1995
THE USE OF CURRICULUM PRESENTATION COMPONENTS	64	64	43	35
THE USE OF LEARNING PROCESS COMPONENTS	56	56	33	38
THE USE OF CONTENT STRUCTURE COMPONENTS	75	75	75	75
THE USE OF VISUAL COMPONENTS	56	51	44	44
THE USE OF VERBAL COMPONENTS	78	88	44	44
TOTAL QUOTIENT OF TEACHING QUALITY	61	61	42	42

Tab.5

Curriculum presentation components are most frequently present in the textbooks by Didaktis and Prometheus (2009). The same applies to verbal coefficients, because it is a new textbook produced after the introduction of Framework Education Programme. In comparison with older textbooks by Prometheus, they contain a summary of the curriculum, charts, diagrams, etc. Very important is the inclusion of a summary of the subject matter, because it helps pupils consolidate, and make global overview and the structure of the curriculum. Only the textbooks published by Prometheus contain a clear explanatory text. Although the textbook does not have the highest proportion of the system presentation curriculum coefficient, it includes schematics and diagrams to simplify the orientation and structure of knowledge. In my point of view, all textbooks could reach even higher coefficients, but they are more thematic and therefore contain a summary of the curriculum for the entire or previous year. As the textbooks contain photos and video presentations, they have a high coefficient of non-verbal components. Therefore the textbooks may be attractive but expensive, too. Only the textbooks by the Prometheus (2009) contain some artwork as non-verbal components. Artistic illustrations complement the historical contexts to enrich the theme, and to motivate students.

Learning process components are most frequently present in the new textbooks by Didaktis and Prometheus (2009). Verbal components are broaden with detailed stimulation such as motivational tasks and other interesting facts from everyday life and historical contexts. Visual components are enriched by using special colours for certain parts of the verbal text. All textbooks use a special font and separate the different parts of the text. Only the textbook by Didaktis contains instructions for working with it. The textbook contains many different structural elements. Pupil can have some difficulties with orientation within the textbook. The textbook by Prometheus for non-technical schools (1996) does not differentiate the curriculum for basic and extended education. Fundamental weakness of all textbooks is the fact that the textbooks do not state learning objectives. The textbooks by Didaktis contain means for students' self-assessment, which is a great advantage. On the other hand, they does not contain results of the tasks and exercises, which I see as a disadvantage. The textbooks published by Prometheus (2009) contain links to other sources of information, which may be more appropriate for talented or more active pupils. Coefficient of learning control system could be a little higher if all the books were not focused thematically, because they contain questions and tasks for the entire year and the questions and tasks for the previous year.

Structural components occur in the same proportion in all the books. This corresponds with the fact that the orientation apparatus contains 4 verbal components only. All textbooks contain content, divided into the chapters, the register and do not contain marginal. The highest coefficient of verbal components can be identified in the latest books by Didaktis (56%). When we compare the older and newer textbooks, we can see the increase in the occurrence of verbal components. But the progress is not as high as in the case of visual components.

The highest representation of visual components can be seen in the textbook by Prometheus (2009) (88%). Based on our initial analysis, it was likely that a higher coefficient of visual components will be in the latest textbook published by Didaktis for its graphic design. The textbook by Prometheus contains higher number of different visual components (e.g. an art reproduction). When comparing the coefficients of verbal and visual components of the older books are balanced ratio of both factors, whereas the newer textbooks greatly outweighs the coefficient representation of visual components.

The newer textbooks by Didaktis and Prometheus (2009) have a higher total coefficient of teaching quality. After the introduction of FEP there was a significant shift in textbook processing. This corresponds with a higher total coefficient of teaching quality of newer textbooks. We can also deduce that textbooks published at the same time have the same coefficient of teaching quality. We formulated a hypothesis that the apparent difference in the overall rate of teaching quality depends on the type of school. However, it was not confirmed.

The following chart (Chart 1) illustrates the proportion of partial and total coefficient of teaching quality. The newer textbooks have the same partial coefficients EI, EII and $EIII$. However, these textbooks have different representation of verbal and visual components. This corresponds with the fact that even though they have the same partial coefficients EI, EII and $EIII$, they have a different representation of occurrence of structural components. This implies that even if the textbooks have the same total coefficient of teaching quality, they have a different structural elements. And thus they fulfil a different function in the educational process. From this we can deduce that it is important to undergo further examining of the textbooks, which will give us a more comprehensive approach.

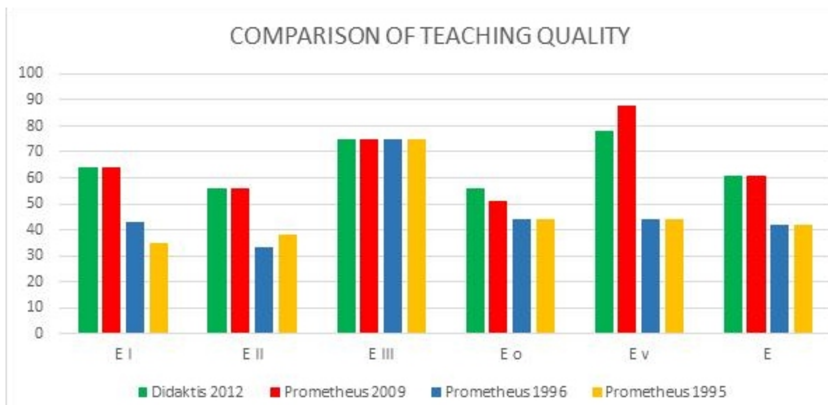

Chart 1

The same conclusion can be derived from two older textbooks, although the conclusion is drawn from the different base. The Textbooks published by Prometheus (1996) and textbooks published by Prometheus (1995), have different coefficients of EI and EII, but on the other hand, coefficients Ev and Eo are the same. It may give us a point that even though both of these books have the same total coefficient of educational quality, they contain other structural components and will once again provide a different function in the educational process.

3. **Final remarks**

From the previous conclusions we can conclude that the educational quality is only a recommended fact for evaluation and analysis of the textbooks. From the comprehensive view, it is necessary to continue with careful examination, which does not ignore the textbooks structure, for example images etc.

The educational quality predetermines the procedural efficiency in the teaching, however, it does not guarantee the fact that the textbooks will be actually used in practice; this statement is based on the fact that the textbooks published by Prometheus (1995) are still used in the teaching process. However, it may be that a consequence of the textbook with higher levels of educational quality are newer and teachers are familiar with them.

The question is, whether the educational quality is still an effective tool for the evaluation of textbooks. Calculation and didactic principle were published by Jan Průcha in 1998. Since then, the concept of the Czech educational system has completely changed. The biggest change took place after the introduction of framework educational programmes, which formed new textbooks. Nowadays, the textbooks are therefore subject of different requirements and it is more than likely that the structural elements of the textbook will change.

Another question might be whether the educational quality is the universal standard for all types of textbooks. This is a significant difference between textbooks and humanistic and technically oriented textbooks. It is evident that teaching style is totally different at technical and humanities schools, it follows that the structure of the textbooks. The structures are dissimilar. From my point of view, I think that the educational quality of textbooks is in favour of humanistic oriented textbooks. It may be derived from the fact that the selected textbooks had a lower rate of the total coefficient of the educational quality.

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ON TEACHING OF GEOMETRIC TRANSFORMATIONS IN SCHOOL

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Abstract

The current core curriculum in mathematics for lower secondary school (3-rd educational level in Poland) omits formal definitions of concepts related to geometric transformations in the plane and is based on their intuitive sense. Practice shows that the current approach makes teaching very difficult and the students solve the typical tasks, not understanding the meaning of geometrical concepts. The article contains basic concepts connected with geometric transformations and examples of geometric tasks that are solved in the third and also in the fourth educational level in an intuitive way, sometimes deviating or even incompatible with the mathematical definition. We show how they could be solved in easier way with introducing definitions of geometric transformations in a simple and understandable for students way sometimes using vector calculus. We take into account isometries: reflection and point symmetry, rotation and translation and similarities with particular consideration on homothetic transformation.

1. Introduction

As the result of permanent reduction of core curriculum in mathematics in past few years (see [8],[9]) and practical attitude to mathematical tasks (an old gymnasium exam: from 2002 to 2011), the big problems with teaching geometric transformations appeared. The three main reasons are:

(1) reduction of contents connected with forming concept of a function;
(2) eliminating vectors;
(3) drastic reduction of contents connected with geometric constructions.

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The above reasons make teaching geometric transformations very difficult. In this case education is based on an intuitive understanding of concepts like symmetry, similarity etc. that may cause numerous misunderstandings. Of course, we do not want to discourage students introducing difficult mathematical formalism but primary theory is also necessary for them. The teachers realize that good theoretical bases help to solve mathematical tasks ([3], [4], [5], [6], [7]). Some of the educators suppose that students of the 3-rd educational level are able to understand geometrical definitions and relations between geometric concepts. According to Dutch psychologist P. M. Van Hiele model of learning geometry by students one can say that pupils of lower secondary school are on level 2 (Abstraction). At this level, properties are ordered. Students understand that properties are related and one set of properties may imply another property ([1]).

The main goal of this article is to remind the most important concepts connected with geometric transformations and discuss the possibility of implementing more theory to lessons instead of using intuitive sense. It is very important now because mathematics became separate subject on the new gymnasium exam (since 2012) and we hope its role will be increasing. Today it is a chance to return to less practical but more mathematical attitude to solving tasks.

2. Geometric transormations

Let us consider the idea of geometric transformation. If we understand it intuitively, it is the change of the location of the points that happens according to some strict rules. So it is a function that relates points of the plane (or space) to the other points of it. In this article we assume traditional definition used in polish school mathematics since the sixties years of the XX century introduced by professor Zofia Krygowska. More precisely:

Definition 1. *Denote as π the arbitrary plane or space. Every bijection $f : \pi \to \pi$ is called the geometric transformation.*

If we accept the above definition it may cause some problems. The function in this definition is not numerical. Domain and range of the function are the sets of points. The students rarely realize that in every geometric transformation the locations of all points change. For example, in school practice they often find only the images of favoured points (vertices of a polygon) forgetting that the other points also change location. Similar approach causes misunderstanding of other important facts. It seems to be obvious that geometric transformations should be taught after the functions and be the excellent example of non-numerical ones. It is also problematic to explain how to understand the word "bijection" in this case. We suggest such an explanation:

(1) for the arbitrary two different points X, Y from π their images $f(X)$, $f(Y)$ are also different,
(2) every point $Y \in \pi$ has to be the image of some point $X \in \pi$ (i.e. $Y = f(X)$ for some point X).

It can be helpful to create some drawing of simple transformation (reflection symmetry) and show that conditions (1) and (2) are satisfied.

3. Vectors and their applications

It is alarming, from the point of teaching geometrical transformations, that vector calculus was eliminated from core curriculum. Vectors make definitions of some geometric transformation simpler and shorter. Vector calculus is also the best and the quickest instrument used in solving mathematical tasks from the analytic geometry. The problem is that today students are supposed to show the result – not to compute it. This way is not correct. First of all, we are not always able to read the result (from the drawing). Secondly, tasks often have more than one solution – vector calculus lets us find all the solutions. Of course, free vector is too complicated question to use at this stage of education but we may identify the vector with its coordinates and then use this helpful tool. Here are the examples of using vector calculus in tasks addressed to students in the third educational level.

Example 1. *Check whether the triangle ABC is right-angled, if $A = (-1, 2)$, $B = (0, 0)$, $C = (6, 3)$.*

Standard analytic solutions are:

- computing distances $d(A, B), d(A, C), d(B, C)$ and checking if the square of one of the distances is equal to the sum of squares of the other distances (Pythagorean theorem) or
- finding the equations of straight lines AB, BC and AC and checking if one of the straight lines is perpendicular to the other one.

But the two above solutions are numerically complicated because they require many various tough calculations. Using vector calculus, we can solve the problem in easy and quick following way:

We compute the coordinates of the vectors $\vec{AB}$, $\vec{BC}$ and $\vec{AC}$:

$$\vec{AB} = [1, -2], \quad \vec{BC} = [6, 3], \quad \vec{AC} = [7, 1].$$

We now notice that $\vec{AB} \circ \vec{BC} = 0$ (scalar product) so these two vectors are perpendicular and finally the triangle ABC is right-angled.

Example 2. *Check if the tetragon $ABCD$ is a rhomboid, if $A = (1, 4)$, $B = (6, 5)$, $C = (56, 105)$, $D = (51, 104)$.*

Standard solutions are:

- computing the distances $d(A, B)$, $d(B, C)$, $d(C, D)$ and $d(D, A)$ and checking if we obtain two pairs of equal distances (more precisely: $d(A, B) = d(C, D)$ and $d(A, D) = d(B, C)$) or
- finding the equations of straight lines AB, BC, CD and AD and checking if we have two pairs of parallel lines.

Of course these two ways are not the fastest methods. Let us notice that:

$$\vec{AB} = \vec{DC} = [5, 1].$$

This fact proves that tetragon $ABCD$ is a rhomboid because the equality of coordinates of vectors causes not only the equality of the lengths of line segments but also their parallelism.

Example 3. *Calculate the area of a triangle ABC, if $A = (1, 6)$, $B = (2, 8)$, $C = (9, -1)$.*

If we try to solve the above problem in the third or fourth educational level we have to face with complicated calculations as standard solutions are:

- calculating the lengths of all sides of the triangle and then use Heron's formula or
- calculating the length of one side of the triangle and then the proper height of a triangle (as a distance between the straight line and the point).

The fastest method (possible even in the third educational level) is to calculate coordinates of two vectors that have common beginning and then use well known formula that lets compute the area of a triangle:

$$\vec{AB} = [1, 2], \vec{AC} = [8, -7].$$

$$P_{ABC} = \frac{1}{2} \cdot \left| \begin{vmatrix} 1 & 2 \\ 8 & -7 \end{vmatrix} \right| = 11.5.$$

The other examples showing applications of vector calculus will be discussed in the sections connected with point symmetry, homothetic transformation and examples of tasks for students.

4. Isometries

Content ISOMETRY is rarely used in school mathematics. If the teachers talk about isometry they explain to the students that it is a geometric transformation that does not change the size and the shape of geometric figures. It is intuitively clear. Of course, isometry can change only the location of the figures: it transforms them into another place or only replaces the points (the whole figure stays in the same place). In addition, only two isometric transformations are discussed: reflection symmetry and point symmetry. There is no information about translation and rotation that is puzzling since particularly translation is one of the most important transformations that is used in the whole fourth educational level when the teachers explain the transformations of the functions' graphs. We think that the definition of an isometry is not too difficult for students. If we define the distance between the points X, Y as follows: $d(X,Y) = \left|\vec{XY}\right|$, we may define the formal but simple definition of an isometry.

4.1. **Formal definition.**

Definition 2. *Geometric transformation $f : \pi \to \pi$ is called an isometry if for the arbitrary points $X, Y \in \pi$ we have the following equality:*

$$d(f(X), f(Y)) = d(X,Y),$$

i.e. the distance between two arbitrary points is equal to the distance between their images.

From the above definition we can notice at once that the sizes of the figures and their shapes cannot change because the distances between proper points stay the same. We now can explain why in isometric transformations we may only find the images of all vertices of a polygon in order to find an image of the whole polygon. Moreover, we also have to discover only an image of the centre of the circle in order to find an image of the whole circle (because the radius stays the same) although above figures contain infinite number of points. Of course, it is not possible for non-regular figures.

In school mathematics we should talk about the following examples of isometric transformations:

(1) REFLECTION SYMMETRY;
(2) POINT SYMMETRY;
(3) TRANSLATION;
(4) ROTATION.

Above transformations will be discussed in the next subsections.

4.2. **Reflection symmetry.** If we introduce the distance between the point X and straight line l as follows:

$$d(X, l) = \inf\{d(X,Y) : Y \in l\}$$

and the following definition:

Definition 3. *$\vec{u} \perp l$ if and only if for every two different points $X, Y \in l$ we have the condition: $\vec{u} \perp \vec{XY}$,*

then we can introduce the following definition.

Definition 4. *Assume that l is any straight line in the plane π. Geometric transformation $f : \pi \to \pi$ is called reflection symmetry towards straight line l if for any point $X \in \pi$ the following conditions are satisfied:*

$$d(X, l) = d(f(X), l), \tag{1}$$

$$\overrightarrow{Xf(X)} \perp l. \tag{2}$$

Let us notice that Definition 4. holds also for $X \in l$. In school practice we modify Definition 4. in the following way:

Definition 5. *Assume that l is any straight line in the plane π and point X does not belong to l. Point X' is symmetric to point X towards l if it satisfies the following conditions:*

- *Distances $d(X, l)$ and $d(X', l)$ are the same,*
- *Line segment XX' is perpendicular to the straight line l.*

In addition we assume that if $X \in l$, then $X' = X$.

Let us notice that Definition 5. has also two conditions, we do not need to add that points X, X' are located on both sides of the straight line l (as it is added in many handbooks).

4.3. **Point symmetry.**

Definition 6. *Assume that S is any point in the plane π. Geometric transformation $f : \pi \to \pi$ is called the point symmetry towards point S if for the arbitrary point $X \in \pi$ the following condition is satisfied:*

$$\vec{XS} = \overrightarrow{Sf(X)}. \tag{3}$$

The above definition holds also for $X = S$.

In school practice we modify Definition 6. in the following way:

Definition 7. *Assume that S is an arbitrary point in the plane π and point X is different from point S. Point X' is symmetric to point X towards point S if it satisfies the following condition: Point S is the centre of the line segment XX'. In addition, if $X = S$ then we assume $X' = X$.*

Let us notice that above definition contains only one condition. We do not have to talk about the same distances and we do not need to add that points X, S, X' are located on the same straight line and points X and X' are different (as it is in many handbooks – then we have three conditions). From Definition 7. we also conclude that vector calculus is the best way of solving tasks connected with point symmetry.

Example 4. *Find an image of point $A = (3, 4)$ in the symmetry towards point $S = (5, 6)$.*

The image of point A can be denoted as $A' = (x, y)$. If we use the definition of point symmetry and vector calculus we obtain:

$\vec{AS} = [5 - 3, 6 - 4] = [2, 2]$,

$\vec{SA'} = [x - 5, y - 6]$.

Because the above vectors are equal we obtain: $x - 5 = 2$ and $y - 6 = 2$. So $x = 7$ and $y = 8$. Then finally $A' = (7, 8)$.

On the base of definitions of reflection and point symmetry we can define the concepts of symmetry axis and symmetry centre of a geometric figure.

Definition 8. *Symmetry axis of a figure is a straight line towards which the figure is symmetric to itself.*

Geometric figures may have no symmetry axis, infinite number of symmetry axes or infinite number of symmetry axes.

Definition 9. *Symmetry centre of a figure is the point towards which the figure is symmetric to itself.*

Geometric figures may have no symmetry centre, one symmetry centre or infinite number of symmetry centres.

Let us notice that point symmetry towards point S is an identical transformation like a U-turn around point S. The above description is the best way to find symmetry centre of a figure. If there exists such a point S around which U-turn gives the same figure (understood as the same set of points) then point S is a symmetry centre of a figure.

The aforementioned concepts have nothing in common. There exist figures that have symmetry centre and have no symmetry axes and figures that have symmetry axes and have no symmetry centre. In addition, we notice that point symmetry can be understood as the superposition of two reflection symmetries in which axes are perpendicular.

4.4. **Translation.**

Definition 10. *Assume that $\vec{u}$ is an arbitrary vector in the plane (or space) π. A geometric translation $f : \pi \to \pi$ is called the translation by a vector $\vec{u}$ if for every point $X \in \pi$ it satisfies the following condition:*

$$\overrightarrow{Xf(X)} = \vec{u}. \tag{4}$$

In school practice we modify the above definition the following:

Definition 11. *Assume that $\vec{u}$ is an arbitrary vector in the plane (or space) π. The image of a point $X \in \pi$ in translation by a vector $\vec{u}$ is the point X' that satisfies the condition:*

$$\vec{XX'} = \vec{u}.$$

4.5. **Rotation.**

Definition 12. *Assume that S is an arbitrary point in the plane π. The rotation around point S of an angle α is a geometric transformation f of the plane π that for every point $X \in \pi$ satisfies the following conditions:*

$$d(X,S) = d(f(X),S), \tag{5}$$

$$|\angle XSf(X)| = \alpha. \tag{6}$$

And in school practice:

Definition 13. *Assume that S is an arbitrary point in the plane π. The image of a point $X \in \pi$ in rotation around point S of angle α is the point X' that satisfies the following conditions:*

$$d(X,S) = d(X',S),$$

$$\left|\angle XSX'\right| = \alpha.$$

Remark. If we accept these definitions 12. and 13. we additionally have to determine in which way we measure the angle (it is a directed angle). In school practice we usually measure the angle anticlockwise.

5. Similarities

Similarities are colloquially understood as transformations that do not change the shapes of the figures but they may change their sizes. Therefore the difference between similarities and isometries is that similarities may not only change the location of a figure but likewise decrease or increase it in a given scale. Of course, similarities do not change the measures of the angles and if the scale is equal to 1, then the similarity is an isometry. The formal definition is not too difficult for students so we can introduce it in the following way.

5.1. **Formal definition.**

Definition 14. *Assume that k is an arbitrary positive number ($k > 0$). Similarity of the scale k is a geometric transformation $f : \pi \to \pi$ that for all points $X, Y \in \pi$ satisfies the following equation:*

$$d(f(X), f(Y)) = k \cdot d(X,Y).$$

Remark. If the figure F is similar to figure G in the scale k then G is similar to F in the scale $\frac{1}{k}$.

5.2. **Homothetic transformation.** Homothetic transformations have not been discussed in lower secondary schools for a long time. It is mainly caused by eliminating vector calculus. It is alarming because homothetic transformations are special cases of similarities that additionally keep the line segments parallelism and they have many various practical applications in various domains (physics, architecture). We also use homothetic transformations in everyday usage to constructional decreasing or increasing figures in given scale. This is the formal definition.

Definition 15. *Assume that k is an arbitrary non-zero number ($k \neq 0$). Homothetic transformation with centre S and ratio k is a geometric transformation $f : \pi \to \pi$ that for every point $X \in \pi$ satisfies the following condition:*

$$\overrightarrow{Sf(X)} = k \cdot \vec{SX}. \tag{7}$$

Definitions of similarity and homothetic transformation are completely different. First of them is based on distances and the second is based on vector calculus. So the next theorem seems to be very interesting for students.

Theorem 1. *Every homothetic transformation with scale k is a similarity of scale $|k|$.*

We present well known proof of this theorem (see [2]). Denote a homothetic transformation of centre S and scale k by f. Let us assume that X, Y are two arbitrary points in the plane (or space) π. Then we have the following equalities, that prove the theorem:

$$d(f(X), f(Y)) = \left|f(X)\vec{f}(Y)\right| = \left|f(\vec{X})S + Sf(\vec{Y})\right| = \left|-Sf(\vec{X}) + Sf(\vec{Y})\right| =$$

$$= \left|-k \cdot \vec{SX} + k \cdot \vec{SY}\right| = \left|k \cdot \vec{XS} + k \cdot \vec{SY}\right| = \left|k \cdot (\vec{XS} + \vec{SY})\right| = \left|k \cdot \vec{XY}\right| =$$

$$= |k| \cdot \left|\vec{XY}\right| = |k| \cdot d(X, Y).$$

6. Examples of tasks for students

In this section we want to present some tasks connected with geometric transformations and their possible solutions. We want to draw your attention to solutions in which we use elements of theory of isometries and similarities as well as vector calculus. These solutions are much shorter, more simple and more intelligible than traditional intuitive ones.

Task 1. *Find the centre of line segment AB, if $A = (3, 4)$, $B = (11, 22)$.*

Solution: In the fourth educational level students know the formula for the centre of line segment but it is a conclusion from vector calculus. We can solve the above task as it follows:

Denote as $S = (x, y)$ the centre of a line segment AB. Then we have:

$$\vec{AS} = [x - 3, y - 4], \vec{SB} = [11 - x, 22 - y].$$

Of course $\vec{AS} = \vec{SB}$, so we obtain equations $x-3 = 11-x$ and $y-4 = 22-y$. Finally we obtain $x = 7, y = 13$ and $S = (7, 13)$.

Task 2. *Give the example of the figure that satisfies all the following conditions:*

a) it has infinite numbers of symmetry axes;
b) it has the symmetry centre that does not belong to this figure;
c) has positive and finite area.

Solution: It is a very interesting task because the only one type of figures satisfies all conditions. These are circle rings.

Task 3. *Assume that* $A = (1, 1)$*,* $B = (3, 7)$*. Find coordinates of the points that divide the line segment* AB *to three equal parts.*

Solution: Here the vector calculus is also the best tool. Let us denote by $S = (x, y)$ and $T = (u, v)$ the searched points. We have

$$\vec{AB} = [2, 6], \vec{AS} = [x - 1, y - 1],$$
$$\vec{AS} = \tfrac{1}{3}\vec{AB} = \left[\tfrac{2}{3}, 2\right].$$

So we obtain $x = 1\frac{2}{3}, y = 3$ and $S = \left(1\frac{2}{3}, 3\right)$. We can find the coordinates of point T in the same way but we also can use the fact that point T is an image of point S in the translation by the vector $\vec{AS}$ so we finally have:

$$T = (u, v) = \left(1\tfrac{2}{3}, 3\right) + \left[\tfrac{2}{3}, 2\right] = \left(2\tfrac{1}{3}, 5\right).$$

If we want to divide a line segment into more parts we proceed analogously.

Task 4. *Find an image of point* $A = (2, 7)$ *in the homothetic transformation of centre* $S = (2, 11)$ *and scale* $k = -\frac{1}{2}$*.*

Solution: Denote by $A' = (x, y)$ the image of a point A in the above transformation. We have:

$$\vec{SA'} = [x - 2, y - 11],$$
$$\vec{SA} = [0, -4].$$

By the definition 14. we get:

$$[x - 2, y - 11] = -\tfrac{1}{2} \cdot [0, -4] = [0, 2].$$

So we finally obtain $x = 2, y = 13$ and $A' = (2, 13)$.

Task 5. *Find the images of the rectangle $ABCD$:*

a) in homothetic transformation with scale $k = 2$ and centre S located inside the rectangle;

b) in homothetic transformation with scale $k = -\frac{1}{4}$ and centre S located outside the rectangle;

c) in homothetic transformation with scale $k = -3$ and centre S being any vertex of the rectangle.

Remark: It is a very important task. Finding the images of figures in homothetic transformations (especially if the scale is negative) makes possible to notice very interesting facts. It can be easily shown (even on drawings) that if the scale k of homothetic transformation is negative then this transformation is a superposition of homothetic transformation of a positive scale $-k$ and point symmetry towards point S (or U-turn). Students are used to positive scale. In homothetic transformation we also have negative scales thus students may find it difficult to understand this fact.

Task 6. *Which sentences are true?*

a) If the figure has symmetry axis it also has the symmetry centre;

b) If the figure has more than one symmetry centre then it has the infinite number of symmetry centres;

c) If the tetragon has a symmetry centre then it is a rhomboid;

d) Every regular polygon has the symmetry centre;

e) Symmetry centre of a figure always belongs to it;

f) If the triangle has more than one symmetry axis then it is regular;

g) Perpendicular bisector of a line segment is the only one symmetry axis of a line segment;

h) Regular pentagon has the symmetry centre.

Remark: Some of those problems are very important. For example regular polygon has the symmetry centre if and only if the number of its sides is even and every tetragon that has a symmetry centre is a rhomboid.

7. Final remarks

If we analyse the above considerations and examples of tasks we can draw the conclusion that introduction the formal definitions of geometric transformations in 3-rd educational level in proper and intelligible way is possible and makes many geometric tasks easy. It is also significant that formal base causes good comprehension of the above geometric concepts hence a chance for easier work in the third and fourth educational level. In addition, it should be emphasized that vector calculus is the best and fastest tool to solve problems from analytic geometry.

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BLENDED LEARNING IN POLISH SCHOOLS

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Abstract

Development in technology, changing labour market and the need for lifelong learning necessitate the evolution in the education of children and youth. Blended learning is about combining the online learning and traditional methods in order to personalize the learning process. The intensive application of this teaching method in the United States seems to make the American students more engaged in solving the tasks assigned to them. It is believed that it has a positive impact on their final exam results. The article describes the basic principles of blended learning from the perspective of the theory of hybrid. It also includes some personal experience of working with this method.

1. Introduction

Recently technology has become a major part of our everyday life. We use computers and other electronic devices for pleasure, work or self-education. In school there are interactive boards, projectors, computers, notebooks and tablets, but teachers' knowledge and preparation for using modern teaching techniques is not always good enough to be sufficiently effective. Otherwise there's a common statement that the old school gives better results so there is no need for a change. In my opinion technology should influence the school not only to make education more interesting but mostly to make it more effective and useful for our growing digital society.

In the beginning, when computers were introduced in schools, there was a lot of researches about using them for mathematical education. The role of computer programs in learning geometry (e.g. Cabri, Geogebra), functions (graphical calculators or plot making programs) or others parts of mathematics was emphasized. Now, when electronic devices are more available for both teachers and students, it's time for considering them as a regular part of modern education. Computer should be used not only as a mere instrument for certain mathematical applications, but they may be utilized for individualization and controlling the learning process. Including

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computers in the teaching process without forsaking traditional teaching methods is possible through **blended learning**. It is a **formal education program in which a student learns partly through online delivery of content and at partly at a supervised classes.** There are four models of blended learning that categorize the majority of blended-learning programs today in the United States of America:

(1) The Rotation mode -l- students rotate on a fixed schedule or at the teacher's discretion between learning modalities, at least one of which is online learning. The Rotation model has four sub-models:
   - **The Station Rotation model** – students rotate within a contained classroom;
   - **The Lab Rotation model** – the rotation occurs between a classroom and a learning lab for online learning;
   - **The Flipped Classroom model** – the rotation occurs between school for face-to-face teacher-guided practice or projects and home or other off-site location for online content and instruction;
   - **The Individual Rotation model** – each student has an individualized play list.

(2) **The Flex model** – students move on an individually customized, fluid schedule among learning modalities where online learning is the backbone of student learning.

(3) **The A La Carte model** – students take one or more courses entirely online and at the same time continue to have actual classroom educational experiences.

(4) **The Enriched Virtual model** – a whole-school experience in which within each course, students divide their time between attending the school and learning remotely using online delivery of content and instruction.

The definition of the blended learning taxonomy I presented above was introduced by Clayton Christensen Institute. The founder, after whom the institute is named, is a Harvard professor and well known management thinker. He created a disruptive innovation theory which has applications in business and economy. As the institute conducts research also in the field of education, they published paper analysing blended learning through the lens of this theory. I will shortly introduce the basis of the presented ideas only in education context and its implications to the school program development.

There are two basic **types of innovations** – **sustaining** and **disruptive** – that follow different trajectories and lead to different results. Sustaining innovations help organizations make better products or services to improve their customer satisfaction. Disruptive innovations do not try to improve products, but they offer a new definition of what's good. Typically, they are simpler and more convenient products which appeal to the new or less demanding customers. In education sustaining innovations are those slow changes which occur in each school every day in order to make teaching more effective. On the other hand, usage of modern teaching methods and strategies; such as e-learning, blended learning, projects, WebQuest or other containing a lot of digital content, demanding team work and information searching skills is disruptive. These methods of education are very interesting for students because they concern everyday problems and give freedom to students. Without going into the details, there are many disadvantages of such methods as well, so that teachers do not choose them very often.

When industries are in the middle of a disruptive change, they often experience a hybrid stage. So do the schools nowadays. In the blended learning method, the Flex, the A La Carte and The Enriched Virtual models are classified as disruptive, while the Rotation models are in the hybrid zone. Keeping in view the technical preparation in Polish school and the teachers' attitude, I think that hybrid models of blended learning are most likely to be introduced to our schools in the near future. My work is to create programs and check if blended teaching can positively impact on the attitude, knowledge and mathematical skills of polish children.

2. Main results

The hybrid models of blended learning are possible to be introduced in polish schools without making huge organizational changes. Below I present the model of the Rotation model in math lessons.

The Station Rotation model is easy to start if the school has at least few mobile platforms; computers, notebooks or tablets or there is a number of desktop computers in the mathematical classroom. Changing modalities during the lesson can be as follow:

- Introduction to the new subject.
- Division into groups.
- Tasks allocation (at least one online).
- Work within groups.
- Students' rotation to another activity.
- Summary of the lesson.

Although the division that is made there does not have to be only group work during the lesson.

For example realizing a subject about equations there might be tasks for three groups as follow:

(1) Solve equations from exercise in your handbook on your own, then check your answers with a friend next to you.
(2) Work in group with your friends trying to arrange equations to the text exercises given by the teacher and write them down in the notebook. Solve those equations as your homework.
(3) Log in into the school learning platform and complete a test named Equations.

For one task each group has about 12 minutes, after that time teacher check students' progress and order a rotation. The move might seem to be an organizational problem, but after establishing rules and the methods, it will become smooth, contributing positively to students' concentration during the actual task.

The Flipped Classroom model might be useful when subject concern practising specific skills. A student learn the theory and solve basic examples when and where he want, basically at home using the online content. During the lesson more complex problems are being solved with the teacher's support. The method is really time-saving and gives opportunity to include more interesting materials in teaching program. Project work can also be done in the Flipped Classroom model. Students look for necessary information at home, mainly by searching the Internet sources that were given to them on the e-learning platform. Then during the lesson they have time for cooperation and project realization with teacher's supervision and complex evaluation.

The Individual Rotation model is difficult to be introduced in schools without major organizational changes. It can be conducted in schools which can provide a computer for each student during the lesson. My proposition for teachers who have students with very different level of mathematical competencies is to have different tasks assigned to each of them depending on their potential. What is almost impossible to do in regular classes, becomes quite simple using e-learning platform. It just needs clear rules and well organization and each student can succeed on their level.

My experiences with blended learning models started with some specially prepared lessons in the primary and the middle school. I present examples of three rotation models being used in different classes.

The Station Rotation model. Subject: Fractions on the number line. Class: 4th grade, primary school.

Firstly, children were divided into groups of four, each student with different level of mathematical competence. The task was to cross-check homework and to make sure that each group member understands the material. It makes weaker students possible to learn with the help of other students in the group, while others repeat important facts and train their skills.

When the teacher recognized an appropriate level of knowledge in every group member, students were asked to do interactive exercises on tablets using the Matlandia program. Otherwise, they were obligated to finish some more tasks on the paper. Playing a mathematical game was a reward for students' cooperation, which reinforce their motivation to help each other.

The Flipped Classroom model. Subject: The history of the numbers. Class: 6th grade of the primary school.

Students did the project on the history of the numbers. The class was divided into five groups, students could group with whoever they wanted. They received particular topics:

- How did the primitive man count?
- The sexagesimal system – is it still actual?
- How was the Ancient Egyptian mathematics?
- The Roman numerals – now and then.
- The Arabic numerals.

All of the materials were accessible on school's e-learning platform, where students discussed about their researches and ideas. Teacher was available on certain time in the chat-room or it was possible to mail her or ask using the forum module. After two weeks of online homework on the project, students began to work during math lessons. They had to cooperate to decide what to do and when or how to work to complete all given tasks.

E-learning platform in that situation facilitated communication and gave better access to all necessary online contents. Each group leader and the teacher could oversee other group members' work and make sure that the task was completed in time.

The Individual Rotation model. Subject: The proportionality and the inverse proportionality. Class: 1st grade of the middle school.

It was a series of lessons about the proportionality, where students had tablets with the Internet access all the time. Instructional films, exercises and tests were available on the school's e-learning platform. Students for three lessons work on their own using online contents delivered at a speed they are comfortable to work with. They decided whether to watch the film with full exercise instruction or not, solving given problems on their own. All solutions had to be written in their notebooks as well. After those lessons and the appropriate quiz, a group of students was picked to receive individual teacher's help. At the same time, others worked on more complex problems expanding their knowledge and skills. Summarizing lessons were conducted in the group working form. That way of teaching gives very interesting and promising results. As the final questionnaire shows, students were very committed to their self-development, they felt responsible for their knowledge. Moreover, they were satisfied because of their achievements, despite not all of them reaching the same level. I think that more researches should be done in using the Individual Rotation model in order to observe whether it was just one time success or the model has a real potential.

In the USA schools which are teaching using blended learning models usually get specially adjusted software that combine e-learning platform functions with professionally prepared interactive contents. In Poland schools usually have to work with free software, especially e-learning platforms like Moodle or Olat which do not always meet all their needs. In addition, there is not much polish mathematical materials available – neither free of charge nor paid ones, but number of them is growing every moment.

Those inconveniences should not stop polish teachers' progress. The traditional, pen-and-paper lectures are not likely to interest students who live in the colourful world with unlimited access to the information by their mobile devices. Teachers have to strive for their attention and try to sell knowledge like products are being sold by professional traders. Even without best materials we can have influence on students learning process so that education will achieve higher level.

3. Final remarks

Poland is prepared for modern teaching methods. The General Statistical Office report declares that more than 90% of primary schools and more than 80% of middle and high schools in Poland are computerized. Depending on the type and the location, one school provides one computer with the Internet access to 10-15 students. It is possible to work with the Rotation model where the internet access is not required for all students in the same time. Implementation of the Flipped Classroom model should not be a problem as well because most of the online learning is done at home on students' personal computers, and each school has a computer laboratory where students who do not have their own devices can work after lessons.

Blended learning, which is very popular in the USA, is now possible to be introduced to polish schools. There are many programs that support process of schools computerization and improve teachers' skills in using electronic devices. If polish teachers open up to the possibilities provided by technology, teaching and learning in polish school will move into a new dimension. It will prepare students well for the age of information and technology.

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Part II
Mathematics and its Application

Scientific Issues
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ON SOME CHARACTERIZATION OF AN INVERSE PROPORTIONALITY TYPE FUNCTION

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Abstract

We deal with a functional equation of the form

$$f(x+y) = F(f(x), f(y))$$

(so called addition formula) assuming that the given binary operation F is associative but its domain is not connected. The aim of the present paper is to discuss solutions of the equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}.$$

It turns out that this functional equation characterized an inverse proportionality type function, but if the domain of the unknown function has no neutral element. In this paper we admit fairly general structure in the domain of the unknown function.

1. Introduction

The functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$$

is the particular case (with $\alpha = 0$) of the equation, which was considered by the author in [3]. In this work for this case we obtain only trivial solutions.

A map $F : \{(x,y) \in \mathbb{R} : \quad x \neq -y\} \longrightarrow \mathbb{R}$ of the form

$$F(u,v) = \frac{uv}{u+v}$$

is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$, that satisfies the equation

$$F(F(x,y),z) = F(x, F(y,z))$$

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for all $(x, y, z) \in \mathbb{R}^3$ such that sums $x + y, y + z, F(x, y) + z, x + F(y, z)$ are not equal to 0. Rational functions with such or similar properties are termed associative operations. The class of the associative operations was described by A. Chéritat [1] and his work was followed by paper [2] of the author.

For every constant $c \in \mathbb{R} \setminus \{0\}$ a homographic function $\varphi : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(x) = \frac{c}{x}, \quad x \neq 0$$

satisfies the functional equation

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}$$

for every pair $(x, y) \in \mathbb{R}^2 \setminus D$, where

$$D = \{(x, -x) : \quad x \in \mathbb{R}\} \cup \{(x, 0) : \quad x \in \mathbb{R}\} \cup \{(0, x) : \quad x \in \mathbb{R}\}.$$

We shall determine all functions $f : G \longrightarrow \mathbb{R}$, where $(G, +)$ is a group, that satisfy the functional equation

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}. \tag{1}$$

A neutral element of the group $(G, +)$ will be written as 0.

By a solution of the functional equation (1) we understand here any function $f : G \longrightarrow \mathbb{R}$ that satisfies equality (1) for every pair $(x, y) \in G^2$ such that $f(x) + f(y) \neq 0$. Thus we deal with the following conditional functional equation:

$$f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)} \tag{E1}$$

for all $x, y \in G$. In [3] we can find the following:

Theorem 1. *Let $(G, +)$ be a group. The only solution $f : G \longrightarrow \mathbb{R}$ of the equation* (E1) *is $f = 0$.*

We want to obtain a non-trivial solution of this equation.

2. Main result

We obtain non constant solutions of the equation (E1) if we consider the unknown function not on the whole group, but on complement of the neutral element. Thus we deal with the following conditional functional equation with a restricted domain:

$$f(x)+f(y)\neq 0 \quad \text{implies} \quad f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)}, \quad x,y,x+y\in G\setminus\{0\}, \tag{E}$$

for functions $f: G\setminus\{0\}\longrightarrow\mathbb{R}$, where $(G,+)$ is a group with the neutral element 0. First we observe that solutions of (E) vanishing at the non-zero element a are trivial.

Lemma. *Let $(G,+)$ be a group and let $f: G\setminus\{0\}\longrightarrow\mathbb{R}$ be a solution of the equation* (E) *such that $f(a)=0$ for some $a\neq 0$. Then*

$$G\setminus\{0,-a\}\subset S\cup(S-a),$$

where

$$S:=\{x\in G\setminus\{0\}: \quad f(x)=0\}.$$

Proof. Let $f: G\setminus\{0\}\longrightarrow\mathbb{R}$ be a solution of the equation (E). On setting $y=a$ in (E) we have

$$f(x)=0 \quad \text{or} \quad f(x+a)=0, \quad x\neq -a, x\neq 0$$

which was to be shown. □

In the light of Lemma, we consider such solutions of (E) which are defined on $G\setminus\{0\}$ and have 0 off their ranges.

Theorem 2. *Let $(G,+)$ be a group. A function $f: G\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$ yields a non-constant solution to the equation* (E) *if and only if there exist a homomorphism $A: G\longrightarrow\mathbb{R}$ such that $0\notin A(G\setminus\{0\})$ and*

$$f(x)=\frac{1}{A(x)}, \quad x\in G\setminus\{0\}.$$

Proof. Let $f: G\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$ fulfils the equation (E). Then

$$f(x)+f(y)\neq 0 \quad \text{implies} \quad \frac{1}{f(x+y)}=\frac{f(x)+f(y)}{f(x)f(y)}=\frac{1}{f(x)}+\frac{1}{f(y)}$$

for $x,y,x+y\neq 0$. This states that a function $g: G\longrightarrow\mathbb{R}$ of the form

$$g(x):=\begin{cases}\frac{1}{f(x)} & \text{for } x\in G\setminus\{0\}\\ \mathit{arbitrary} & \text{for } x=0.\end{cases}$$

yields a solution of the equation

$$g(x)+g(y)\neq 0 \quad \text{implies} \quad g(x+y)=g(x)+g(y) \quad x,y,x+y\neq 0.$$

We omit here a simple calculation showing that $f(x)+f(y)=0$ if and only if $g(x)+g(y)=0$, for all $x,y\in G\setminus\{0\}$.

From the theorems proved by R. Ger [5], J. G. Dhombres, R. Ger [4] we conclude that there exist a homomorphism $A : G \longrightarrow \mathbb{R}$ groups $(G, +)$ and $(\mathbb{R}, +)$ such that

$$g(x) = A(x), \quad x \in G \setminus \{0\}.$$

Clearly, $0 \notin A(G \setminus \{0\})$ and

$$\frac{1}{f(x)} = A(x) \quad \text{for} \quad x \neq 0,$$

whence

$$f(x) = \frac{1}{A(x)}, \quad x \neq 0.$$

It is easy to check that the above formula establishes a solution of the equation (E). Thus the proof has been completed. □

Remark. *Continuous solutions of the equation* (E) *are inverse proportions.*

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FUZZIFIED PROBABILITY: FROM KOLMOGOROV TO ZADEH AND BEYOND

ROMAN FRIČ, MARTIN PAPČO

Abstract

We discuss the fuzzification of classical probability theory. In particular, we point out similarities and differences between the so-called fuzzy probability theory and the so-called operational probability theory.

1. Introduction

Clearly, the classical (Kolmogorovian) probability theory ([12]), CPT for short, based on Boolean logic and set theory, has its limitations when modelling real life situations and when uncertainty is to be taken into account. Accordingly, L. A. Zadeh ([19]) proposed to extend random events to fuzzy random events. In the literature, there are various approaches to uncertainty based on fuzzy sets and fuzzy logic. Also, there are several generalizations of CPT and we will deal with two of them, both deserving to be called fuzzified probability theory; just to avoid misunderstanding, we call them *fuzzy probability theory*, FPT for short, and *operational probability theory*, OPT for short. While FPT is more oriented to applications and engineering and its basic ideas and constructions are outlined in [1], OPT originated in modelling quantum phenomena in physics, but its mathematical results can be used, e.g., in social sciences, and the interested reader is referred to [11], [2], [3], [7], and references therein.

We aim at a better understanding of the fuzzification of CPT. Since we will work with more than one probability theory, we need to put them into

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a perspective. To do so, we first analyse and discuss some basic notions and constructions of probability theory.

Our paper is written as a series of questions and answers. This way we hope to provide more information (stress is on "why" and "what") within a limited space and make the presentation more readable.

2. Basic notions of CPT and their roles

Probability theory has been axiomatized (mathematized) by A. N. Kolmogorov. The influence of his "Grundbegriffe" ([12]) on the education in the area of probability and its impact on research in stochastics cannot be overestimated. In this section we present few remarks about basic mathematical notions of CPT: *random events, probability measures, and random variables.*

At the beginning we have a *probability space* $(\Omega, \mathbf{A}, P)$, where Ω is the set of all outcomes of a random experiment, $\mathbf{A}$ is a σ-field of subsets of Ω, each $A \in \mathbf{A}$ is called an *event*, events of the form $A = \{\omega\}$, $\omega \in \Omega$, are called elementary events, and $P: \ \mathbf{A} \to [0,1]$ is a normed σ-additive measure called *probability*; $P(A)$ measures how "big" is $A \in \mathbf{A}$ in comparison to Ω. The most important example is $(R, \mathbf{B}_R, p)$, where R are the real numbers, $\mathbf{B}_R$ is the real Borel σ-field, and p is a probability on $\mathbf{B}_R$. Let f be a measurable map of Ω into R, i.e., $f^{\leftarrow}(B) = \{\omega \in \Omega \mid f(\omega) \in B\} \in \mathbf{A}$ for all $B \in \mathbf{B}_R$. If $p(B) = P(f^{\leftarrow}(B))$ for all $B \in \mathbf{B}_R$, then f is said to be a *random variable* and p is said to be the distribution of f. More generally, if $(\Xi, \mathbf{B})$ is a measurable space then a measurable map $g : \Omega \longrightarrow \Xi$ is said to be a $(\Xi, \mathbf{B})$-valued random variable. If points of Ω and Ξ are viewed as the degenerated probability point-measures, then g gives rise to a map T_g on the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$, $\Omega \subset \mathcal{P}(\mathbf{A})$, T_g and g coincide on Ω.

Remark 2.1. *Each random event $A \in \mathbf{A}$ can be viewed as the indicator (characteristic) function χ_A of A, or as a propositional function "$\omega \in A$" and operations on events correspond to (Boolean) logical operations on propositional functions. Observe that P can be viewed as a fuzzy subset of $\mathbf{A}$.*

Remark 2.2. *A random variable is neither a variable, nor random. Indeed, f is a function and the assignment $\omega \mapsto f(\omega)$ is not random. Instead, for example, if $F : R \to [0,1]$ is a distribution function corresponding to p, then the pair (R, F) looks like "randomized real variable". In probability theory and mathematical statistics, in fact, laws of probability on R, R^n, or R^R (in terms of distribution functions, characteristic functions, density functions) are the main objects of study and the random variables, as measurable functions, play only an auxiliary role.*

Remark 2.3. *In some sense, more important than a random variable f is its dual preimage map $f^{\leftarrow}$, mapping $\mathbf{B}_R$ into $\mathbf{A}$. Indeed, $f^{\leftarrow}$ "preserves" the Boolean structure of $\mathbf{B}_R$, i.e. it is a Boolean homomorphism mapping real events $\mathbf{B}_R$ into the events of the original (sample) space $(\Omega, \mathbf{A}, P)$, and p is the composition $P \circ f^{\leftarrow}$ of $f^{\leftarrow}$ and P. The stochastic information about an "observed" event $B \in \mathbf{B}$ is obtained by finding the corresponding "theoretical" event $f^{\leftarrow}(B) \in \mathbf{A}$ and $P(f^{\leftarrow}(B))$ is the needed stochastic information about B. In CPT, strange enough, $f^{\leftarrow}$ does not have its name.*

Remark 2.4. *Let f be a measurable map of Ω into R. Then f induces a map D_f of the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B}_R)$ of all probability measures on $\mathbf{B}_R$: for $Q \in \mathcal{P}(\mathbf{A})$ we define*

$$D_f(Q) = Q \circ f^{\leftarrow};$$

we say that f pushes forward Q to $D_f(Q)$. If we identify each $\omega \in \Omega$ and the Dirac point-probability δ_ω and, similarly, each $r \in R$ and δ_r, then a straightforward calculation shows that $D_f(\delta_\omega) = \delta_{f(\omega)}$. Consequently, the distribution map D_f can be considered as an extension of f, mapping $\Omega \subseteq \mathcal{P}(\mathbf{A})$ into $R \subseteq \mathcal{P}(\mathbf{B}_R)$, to D_f mapping $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B}_R)$.

QUESTION 1. *What is the role of a random variable in* CPT?

ANSWER 1. *It is a channel through which stochastic information is transported (the dual preimage map transports the real random events $\mathbf{B}_R$ into the sample random events $\mathbf{A}$ and each probability measure P on $\mathbf{A}$ is transported via the composition of the preimage map and P to become a probability measure on $\mathbf{B}_R$).*

3. Basic notions of FPT and their roles

In order to understand the transition from CPT to FPT, we first recollect some facts about FPT. The next lines are borrowed from [1]:

> "The development of fuzzy probability theory was initiated by H. Kwakernaak ([13]) with the introduction of fuzzy random variables in 1978... Fuzzy probability theory is an extension of probability theory to dealing with mixed probabilistic/non-probabilistic uncertainty... If a set of uncertain perceptions of a physical quantity is present in the form of a random sample, then the overall uncertainty possesses a mixed probabilistic/non-probabilistic character. Whilst the scatter of the realizations of the physical quantity possesses a probabilistic character (frequentative or subjective), each particular realization from the population may, additionally, exhibit non-probabilistic uncertainty. Consequently,

> a realistic modelling in those cases must involve both probabilistic and non-probabilistic uncertainty. This modelling without distorting or ignoring information is the mission of fuzzy probability theory. A pure probabilistic modelling would introduce unwarranted information in the form of a distribution function that cannot be justified and Fuzzy Probability Theory would thus diminish the trustworthiness of the probabilistic results."

QUESTION 2. *What is the meaning of "fuzzy probability law"?*

ANSWER 2. *A classical distribution function F represents a classical probability law and its fuzzification $\tilde{F}$, called fuzzy distribution function, represents a "fuzzy probability law". The fuzzification is based on a suitable non-probabilistic procedure transforming a classical probability space $(R, \mathbf{B}_R, F)$ into $(R, \mathbf{B}_R, \tilde{F}) \equiv (R, \mathbf{B}_R, \mu, F)$, where μ is a membership function constructed via a map of Ω into the fuzzy real numbers (satisfying certain technical conditions) and $\tilde{F}$ represents the fuzzy set of distribution functions determined by μ.*

Remark 3.1. *Observe that random events in CPT and FPT are crisp sets forming a σ-field of sets. The transition from CPT to FPT is based on the transition from F to $\tilde{F}$.*

QUESTION 3. *What is the role of fuzzy random variables?*

M. R. Puri and D. A. Ralescu ([18]) formalized fuzzy random variables (also called random fuzzy sets) as an extension of random sets as follows. Let B be a separable Banach space. Denote $\mathcal{K}(B)$ the set of all non-empty bounded closed subsets of B and denote $\mathcal{F}(B)$ the class of the normal upper semi-continuous $[0,1]$-valued functions defined on B with bounded closure of the support. Let $(\Omega, \mathbf{A}, P)$ be a probability space. A *fuzzy random variable* is a mapping $X : \Omega \longrightarrow \mathcal{F}(B)$ such that, for all $\alpha \in [0,1]$, the set-valued α-level mapping $X_\alpha : \Omega \longrightarrow \mathcal{K}(B)$, $X_\alpha(\omega) = (X(\omega))_\alpha$ is a compact random set, that is, it is Borel measurable with respect to the Borel σ-field generated by the topology associated with the well-known Hausdorff metric on $\mathcal{K}(B)$.

The usefulness of this somewhat technical notion follows, e.g., from a detailed discussion in [4] explaining how, using the notion of fuzzy random variable as a tool, classical methods of mathematical statistics can be modified to get corresponding methods of "fuzzy statistics".

ANSWER 3. *Fuzzy random variables serve as a (non-random) tool to transform classical probability law (in terms of distribution functions) into fuzzy probability law (in terms of fuzzy distribution functions)*

4. Basic notions of OPT and their roles

In OPT (cf. [11], [2], [3]) random events (called also effects) are measurable fuzzy sets and fuzzified probability measures (also called states) are integrals with respect to probability measures — this is in accordance with Zadeh's proposal ([19]). The essential novelty is in the fuzzification of random variables: *the outcome ω of a random experiment can be mapped to a probability measure on* $\mathbf{B}_R$ (or on some other target measurable space)! Accordingly, the measurability of operational random variable (also statistical map, or a fuzzy random variable in the Bugajski-Gudder sense) and the corresponding probability distribution have to be defined in a different way than in CPT. This leads to different interpretation, mathematical apparatus, and applications (cf. [5], [14], [15], [16], [17], [10]).

QUESTION 4. *What is the role of operational random variables*?

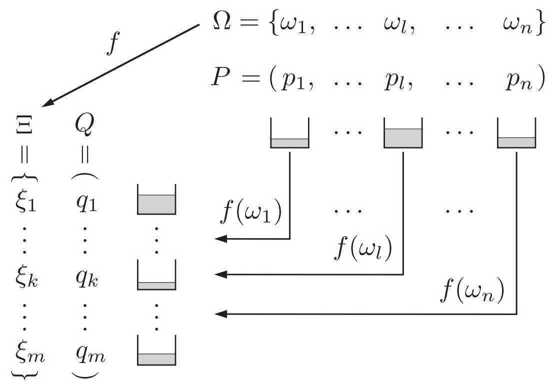

Figure 1

To avoid technicalities, we illustrate the involved notions and constructions on the discrete (finite) case (cf. [9]). So, assume that

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$$

and

$$\Xi = \{\xi_1, \xi_2, ..., \xi_m\}$$

are two finite sets, $\mathbf{A}$ and $\mathbf{B}$ are the sets of all subsets of Ω and Ξ, and P and Q are probability measures on $\mathbf{A}$ and $\mathbf{B}$, respectively. Then P and Q reduce to probability functions $P = (p_1, p_2, ..., p_n)$, $p_l = P(\{\omega_l\})$, $l = 1, 2, ..., n$, and $P(A) = \sum_{\omega_l \in A} p_l$, $A \subseteq \Omega$, resp. $Q = (q_1, q_2, ..., q_m)$, $q_k = Q(\{\xi_k\})$, $k = 1, 2, ..., m$, and $Q(B) = \sum_{\xi_k \in B} q_k$, $B \subseteq \Xi$. Each map $f : \Omega \longrightarrow \Xi$ is measurable and f is a $((\Xi, \mathbf{B})$-valued, classical) random variable whenever $Q(B) = P(f^{\leftarrow}(B))$, where $q_k = Q(\{\xi_k\}) = \sum_{\omega_l \in f^{\leftarrow}(\{\xi_k\})} p_l$

and $Q(B) = \sum_{\xi_k B} q_k = \sum_{\omega_l \in f^{\leftarrow}(B)} p_l$. Simply, the random variable f can be viewed as a system of pipelines through which the probability measure P is distributed to become Q, see Fig. 1. Observe that each p_l goes exactly to one ξ_k.

An operational random variable can be viewed as a more complex system of fuzzified pipelines, see Fig. 2. Each p_l is distributed along Ξ via the probability function (measure) $(w_{1l}, w_{2l}, ..., w_{ml})$.

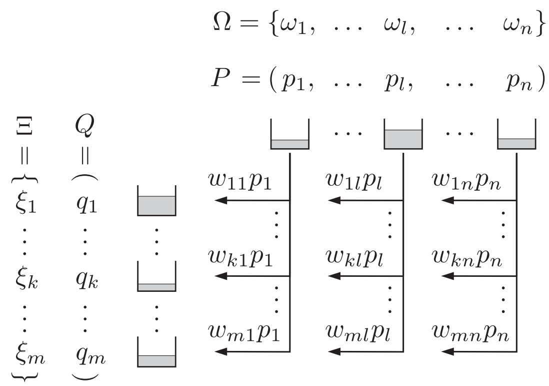

FIGURE 2

For each $l \in \{1, 2, ..., n\}$, p_l is distributed among the elements of Ξ as follows: $w_{kl}p_l$ goes to ξ_k, $k \in \{1, 2, ..., m\}$, and the sum $\sum_{l=1}^{n} w_{kl}p_l$ represents the total input q_k of the probability which flows into ξ_k. Further, for $B \subseteq \Xi$, the sum $\sum_{\xi_k \in B} q_k$ represents the probability $Q(B)$ and, explicitly,

$$Q(B) = \sum_{\xi_k \in B} \sum_{l=1}^{n} w_{kl}p_l = \sum_{l=1}^{n} p_l \sum_{\xi_k \in B} w_{kl}.$$

This points to a fuzzy event $u_B \in [0,1]^{\Omega}$ defined by $u_B(\omega_l) = \sum_{\xi_k \in B} w_{kl}$, $l \in \{1, 2, ..., n\}$, and to a canonical map h_c (defined by $h_c(\chi_B) = u_B$) sending crisp events $\mathbf{B}$ to fuzzy events $[0,1]^{\Omega}$.

The importance of h_c comes from the fact, that $Q(B)$ is "the integral of u_B with respect to P"!

Finally, h_c can be canonically extended to a map h on fuzzy events $[0,1]^{\Xi}$ into fuzzy events $[0,1]^{\Omega}$ so that "the integral with respect to Q" is the composition of h and "the integral with respect to P"!!

Next, we briefly outline how the discrete case of OPT can be extended to the general case.

Remark 4.1. *Let* $(\Omega, \mathbf{A})$ *be a measurable space (remember, we consider subsets as crisp fuzzy sets). Denote* $\mathcal{M}(\mathbf{A})$ *the set of all measurable functions* *of* Ω *into the unit interval* $[0,1]$. *In OPT the operations on fuzzy random events* $\mathcal{M}(\mathbf{A})$ *(generalizations of Boolean operations on classical random events) follow the Łukasiewicz logic:* $x \oplus y = \min\{1, x+y\}$, $x^c = 1 - x$, $x \odot y = \max\{0, x+y-1\}$ *for the unit interval* $[0,1]$, *and coordinate-wise for fuzzy sets. Observe that if* $\mathbf{T} = \{\emptyset, \{a\}\}$ *is a two-element field of sets, then* $[0,1] \equiv \mathcal{M}(\mathbf{T})$.

Remark 4.2. *Let* X *be a set and let* $[0,1]^X$ *be the set of all fuzzy subsets of* X *carrying the coordinate-wise partial ordered (*$v \leq u$ *whenever* $v(x) \leq u(x)$ *for all coordinates* x*) and the partial binary operation of difference "*$\ominus$*" defined coordinate-wise:* $(u \ominus v)(x) = u(x) - v(x)$ *whenever* $v(x) \leq u(x)$ *for all coordinates* x. *In OPT, an important role is played by* D-posets of fuzzy sets, *i.e., subsets* $\mathcal{X} \subseteq [0,1]^X$ *carrying the inherited coordinate-wise partial ordered, containing the top element* 1_X, *the bottom element* 0_X, *and closed with respect to the inherited partial binary operation of difference. Both* $\mathbf{A}$ *and* $\mathcal{M}(\mathbf{A})$, $\mathbf{A} \subset \mathcal{M}(\mathbf{A})$, *are distinguished D-posets of fuzzy sets: they model random events in CPT and in OPT, respectively. Let* h *be a map on a D-poset of fuzzy sets* $\mathcal{Y} \subseteq [0,1]^Y$ *into a D-poset of fuzzy sets* $\mathcal{X} \subseteq [0,1]^X$. *If* h *preserves the order, the top and bottom elements, and the difference, then it is called a* D-homomorphism. *Sequentially continuous (with respect to the coordinate-wise convergence of sequences) D-homomorphisms play a crucial role in OPT (cf. Lemma 4.1 and Corollary 4.2 in* [6]*).*

Theorem 4.3. *Let* $(\Omega, \mathbf{A})$ *and* $(\Xi, \mathbf{B})$ *be measurable spaces.*

(i) *Each sequentially continuous D-homomorphism on* $\mathbf{B}$ *into* $\mathcal{M}(\mathbf{A})$ *can be uniquely extended to a sequentially continuous D-homomorphism on* $\mathcal{M}(\mathbf{B})$ *into* $\mathcal{M}(\mathbf{A})$*;*

(ii) *Integrals on* $\mathcal{M}(\mathbf{A})$ *with respect to probability measures on* $\mathbf{A}$ *are exactly sequentially continuous D-homomorphism on* $\mathcal{M}(\mathbf{A})$ *into* $[0,1]$.

Remark 4.4. *Observe that, according to the Lebesgue Dominated Convergence Theorem, every integral with respect to a probability measure is sequentially continuous. It is easy to verify that a composition of two sequentially continuous D-homomorphisms is a sequentially continuous D-homomorphism. Consequently, a composition of a sequentially continuous D-homomorphism on* $\mathcal{M}(\mathbf{B})$ *into* $\mathcal{M}(\mathbf{A})$ *and a sequentially continuous D-homomorphism on* $\mathcal{M}(\mathbf{A})$ *into* $[0,1]$ *is an integral on* $\mathcal{M}(\mathbf{B})$ *with respect to a probability measure on* $\mathbf{B}$.

Definition 4.5. *Let* $(\Omega, \mathbf{A})$, $(\Xi, \mathbf{B})$ *be measurable spaces. Let* T *be a map on* $\mathcal{P}(\mathbf{A})$ *into* $\mathcal{P}(\mathbf{B})$ *such that, for each* $B \in \mathbf{B}$, *the assignment* $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ *yields a measurable map on* Ω *into* $[0,1]$ *and*

$$\text{(BG)} \qquad \big(T(P)\big)(B) = \int \big(T(\delta_\omega)\big)(B)\, \mathrm{d}P$$

for all $P \in \mathcal{P}(\mathbf{A})$ and all $B \in \mathbf{B}$. Then T is said to be a operational random variable *(also a statistical map, or a fuzzy random variable in the sense of Bugajski and Gudder).*

Remark 4.6. *The assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ results in a sequentially continuous D-homomorphism h_c on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$, hence can be uniquely extended to a sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$. It is known that this way to each operational random variable T of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ there corresponds a unique sequentially continuous D-homomorphism $h = T^{\triangleleft}$ on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ called* observable *and, vice versa (via the composition of a sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ and integrals in (BG)), to each sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ there corresponds a unique T and $h = T^{\triangleleft}$.*

ANSWER 4. *It is a channel through which a fuzzified stochastic information is transported. Each outcome of a random experiment is mapped to a "local" probability measure (possibly a degenerated point-measure, always in CPT) which fuzzifies the corresponding observation and each probability measure P on the sample random events $\mathbf{A}$, and hence the corresponding integral on $\mathcal{M}(\mathbf{A})$ (taking into the account all "local" probability measures) is then transformed to a fuzzified "global" probability measure Q on the observed crisp random events $\mathbf{B}$, and hence the corresponding integral on the observed fuzzy random events $\mathcal{M}(\mathbf{B})$. This yields an operational random variable $T : \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$. To T there corresponds a unique observable $T^{\triangleleft} : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ and the composition of $T^{\triangleleft}$ and the integral on $\mathcal{M}(\mathbf{A})$ with respect to P is the integral on $\mathcal{M}(\mathbf{B})$ with respect to Q.*

5. CPT embedded in OPT

Since to each σ-field $\mathbf{A}$ there corresponds exactly one D-poset of fuzzy sets $\mathcal{M}(\mathbf{A})$ and to each probability measure P on $\mathbf{A}$ there corresponds exactly one integral on $\mathcal{M}(\mathbf{A})$ with respect to P, the extension of CPT to OPT looks "conservative". The real novelty is revealed when comparing the corresponding notions of a random variable.

QUESTION 5. *What is the difference between random variables and observables in CPT and OPT?*

ANSWER 5. *Both a classical random variable and a operational random variable model channels through which probability measures are transported, but the latter has a quantum character: a degenerated probability point-measure can be mapped to a non-degenerate probability measure. Dually, both a classical observable and a fuzzy observable map events (represented* *by special D-posets of fuzzy sets) to events, but the latter can map a crisp event to a genuine fuzzy one.*

QUESTION 6. *How are CPT and OPT related?*

ANSWER 6. *CPT can be embedded into OPT in a natural way: classical random evens form a σ-field $\mathbf{A}$ and are embedded into the fuzzy random events $\mathcal{M}(\mathbf{A})$, every generalized probability measure is an integral on $\mathcal{M}(\mathbf{A})$ with respect to a classical probability measure on $\mathbf{A}$, each classical probability measure is the restriction of such integral to $\mathbf{A}$, and there is a one-to-one correspondence between random events (σ-fields of sets) and fuzzy random events (measurable maps into $[0,1]$). Since every sequentially continuous D-homomorphism on $\mathbf{A}$ into $[0,1] \equiv \mathcal{M}(\mathbf{T})$ can be uniquely extended to an observable on $\mathcal{M}(\mathbf{A})$ to $\mathcal{M}(\mathbf{T})$, to every probability measure P on crisp events $\mathbf{A}$ there corresponds a unique observable (integral with respect to P) on fuzzy events $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$. Consequently, generalized probability measures in OPT become observables. Hence probability measures in CPT are "shadows" of fuzzy morphisms.*

Additional information on categorical approach to probability theory can be found in [7], [8].

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BOUNDARY VALUE PROBLEMS FOR POISSON INTEGRALS FOR HERMITE EXPANSIONS

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Abstract

The aim of this paper is the study the Poisson integral for Hermite expansions. We present some boundary value problems related to this integral and its various modifications.

1. Introduction

Let $L^p(\mathbb{R})$ denote the set of functions f defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \quad \text{if } 1 \le p < \infty,$$

and f is bounded almost everywhere on $\mathbb{R}$ if $p = \infty$.

In the paper [4] the author presented some approximation properties of the Poisson integral for Hermite function expansions given by

$$A(f)(r,y) = A(f;r,y) = \int_{-\infty}^{\infty} r^{\frac{1}{2}} K(r,y,z)\, f(z)\, dz, \quad f \in L^p(\mathbb{R}),$$

where

$$K(r,y,z) = \sum_{n=0}^{\infty} r^n h_n(y)\, h_n(z), \quad 0 < r < 1,$$

$$h_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x)$$

and H_n is the nth Hermite polynomial (see, for example, [10]). The operator $A(f)$ is linear and positive. Basic facts on positive linear operators and its applications can be found in [1, 2].

In [4] the following theorem was proved.

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Theorem 1. [4] *Let* $y_0 \in \mathbb{R}$ *and let* $f = f_1 + f_2$, *where* $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^\infty(\mathbb{R})$. *If* f *is continuous at* y_0, *then*

$$\lim_{(r,y)\to(1^-,y_0)} A(f;r,y) = f(y_0).$$

Gosselin and Stempak in [3] considered the integral $A_0(f)$ of a function $f \in L^p(\mathbb{R})$ defined by

$$A_0(f)(x,y) = A_0(f;x,y) = \int_{-\infty}^{\infty} P(x,y,z) f(z)\, dz,$$

where

$$P(x,y,z) = \sum_{n=0}^{\infty} h_n(y)\, h_n(z) \exp\left(-(2n+1)x\right), \quad x > 0$$

and

$$P(x,y,z) = e^{-x} K(e^{-2x}, y, z).$$

Gosselin and Stempak [3] obtained the following results.

Theorem 2. [3] *If* $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, *then* $A_0(f)$ *is of the class* C^∞ *on the set* $(0,\infty)\times\mathbb{R}$ *and* $A_0(f)$ *is a solution of the differential equation*

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial y^2} - y^2 u(x,y).$$

Theorem 3. [3] *Let* $f \in L^p(\mathbb{R})$. *Then*

(a) $\|A_0(f;x,\cdot)\|_p \le (\cosh 2x)^{-\frac{1}{2}} \|f\|_p, \quad 1 \le p \le \infty,$
(b) $\|A_0(f;x,\cdot) - f(\cdot)\|_p \to 0$ *as* $x \to 0, \quad 1 \le p < \infty,$
(c) $\lim_{x\to 0} A_0(f;x,y) = f(y)$ *almost everywhere,* $1 \le p < \infty.$

It is worth to mention that approximation properties of various Poisson integrals associated with Hermite and Laguerre polynomials were studied in one and two dimensions in [5, 6, 7, 8, 9, 11].

In this paper we indicate boundary value problems related to $A(f)$ and some modifications of this operator.

2. Boundary value problems

Below we present announced theorems. We omit the proofs of them, because there are a simple consequence of previous properties.

Theorem 4. *Let* $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$. *Then* $A(f)$ *is of the class* C^∞ *on the set* $(0,1)\times\mathbb{R}$. *Moreover,* $A(f)$ *is a solution of the problem*

$$-2r\frac{\partial u(r,y)}{\partial r} = \frac{\partial^2 u(r,y)}{\partial y^2} - y^2 u(r,y), \qquad (r,y) \in (0,1)\times\mathbb{R},$$

$$\lim_{r\to 1^-} \|u(r,\cdot) - f(\cdot)\|_p = 0, \qquad 1 \le p < \infty.$$

We introduce the operator A_1 given by

$$A_1(f)(t,y) = A_1(f;t,y) = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{t}\right) K\left(\exp\left(-\frac{2}{t}\right), y, z\right) f(z)\,dz$$

for $f \in L^p(\mathbb{R}),\ 1 \le p \le \infty,\ t > 0$ and $y \in \mathbb{R}$.

Theorem 5. *Let $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$. Then $A_1(f)$ is of the class C^∞ on the set $\mathbb{R}_+ \times \mathbb{R}$ and $A_1(f)$ is a solution of the problem*

$$-t^2 \frac{\partial u(t,y)}{\partial t} = \frac{\partial^2 u(t,y)}{\partial y^2} - y^2 u(t,y), \qquad (t,y) \in \mathbb{R}_+ \times \mathbb{R},$$

$$\lim_{t\to\infty} \|u(t,\cdot) - f(\cdot)\|_p = 0, \qquad 1 \le p < \infty.$$

Let us consider the operator A_2 defined by

$$A_2(f)(r,y) = A_2(f;r,y) = \rho(r) \int_{-\infty}^{\infty} K(r,y,z) f(z)\,dz$$

for $f \in L^p(\mathbb{R}),\ 1 \le p \le \infty,\ 0 < r < 1$, where the function ρ is continuously differentiable in $(0,1)$ and such that

$$\rho(r) > 0 \quad \text{and} \quad \lim_{r\to 1^-} \rho(r) = 1.$$

We introduce the notation

$$T = \frac{\partial^2}{\partial y^2} - y^2 + 2r\frac{\partial}{\partial r} - 2r\frac{\rho'(r)}{\rho(r)} + 1 \quad \text{and} \quad T^2 = T(T).$$

Theorem 6. *Let $y_0 \in \mathbb{R}$. If f is as in Theorem 1, then $A_2(f)$ is of the class C^∞ on the set $(0,1) \times \mathbb{R}$ and $A_2(f)$ is a solution of the problem*

$$Tu(r,y) = 0, \quad (r,y) \in (0,1) \times \mathbb{R},$$

$$\lim_{(r,y)\to(1^-,y_0)} u(r,y) = f(y_0).$$

For $f, g \in L^p(\mathbb{R}),\ 1 \le p \le \infty$ we define the operator V:

$$V(f,g)(r,y) = V(f,g;r,y) = \rho_1(r) A_2(f;r,y) + A_2(g;r,y),$$

where the function ρ_1 is continuously differentiable in $(0,1)$, $0 < r < 1$, $y \in \mathbb{R}$.

Theorem 7. *Let $y_0 \in \mathbb{R}$. If f, g are as in Theorem 1 and*

$$\lim_{r\to 1^-} \rho_1(r) = 0, \quad \lim_{r\to 1^-} \rho_1'(r) = \frac{1}{2}, \quad \frac{\partial}{\partial r}\left(r\rho_1'(r)\right) = 0,$$

then $V(f,g)$ is of the class C^∞ on the set $(0,1)\times\mathbb{R}$ and $V(f,g)$ is a solution of the problem

$$\begin{gathered} T^2u(r,y) = 0, \quad (r,y) \in (0,1)\times\mathbb{R}, \\ \lim_{(r,y)\to(1^-,y_0)} u(r,y) = g(y_0), \\ \lim_{(r,y)\to(1^-,y_0)} Tu(r,y) = f(y_0). \end{gathered}$$

Remark 1. *From the assumptions of Theorem 7 it follows that $\rho_1(r) = \frac{1}{2}\ln r$. In this case the operator V is of the form*

$$V(f,g;r,y) = \frac{1}{2}\rho(r)\ln r \int_{-\infty}^{\infty} K(r,y,z)f(z)\,dz + \rho(r)\int_{-\infty}^{\infty} K(r,y,z)g(z)\,dz$$

for $0 < r < 1$, $y \in \mathbb{R}$.

Theorem 8. *Let $y_0 \in \mathbb{R}$. If f, g are as in Theorem 1 and if*

$$\begin{gathered} \lim_{r\to 1^-} \rho_1(r) = 0, \quad \lim_{r\to 1^-} \rho_1'(r) = \frac{1}{2}, \\ 2r\frac{\partial}{\partial r}\left(r\rho_1'(r)\right) + r\rho_2(r)\rho_1'(r) = 0, \end{gathered}$$

where ρ_2 is some continuous function, then $V(f,g)$ is of the class C^∞ on the set $(0,1)\times\mathbb{R}$ and $V(f,g)$ is a solution of the problem

$$\begin{gathered} T^2u(r,y) + \rho_2(r)Tu(r,y) = 0, \quad (r,y) \in (0,1)\times\mathbb{R}, \\ \lim_{(r,y)\to(1^-,y_0)} u(r,y) = g(y_0), \\ \lim_{(r,y)\to(1^-,y_0)} Tu(r,y) = f(y_0). \end{gathered}$$

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A NOTE ON WEAKLY ϱ-UPPER CONTINUOUS FUNCTIONS

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Abstract

In the article we present definition and some properties of weakly ϱ-upper continuous functions. We find maximal additive and maximal multiplicative families for the class of weakly ϱ-upper continuous functions.

1. Preliminaries

In the article we apply standard symbols and notations. By $\mathbb{R}$ we denote the set of all real numbers, by $\mathbb{N}$ we denote the set of all positive integers. By $\mathcal{L}$ we denote the family of Lebesgue measurable subsets of the real line. The symbol $\lambda(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$. In the whole article, I will denote an open interval (not necessarily bounded) with ends a, b and f – a real function defined in I. By $\mathcal{A}$ we denote the class of all approximately continuous functions defined in I.

Let E be a measurable subset of $\mathbb{R}$ and x be a real number. According to [1], the numbers

$$\overline{d}^{+}(E,x) = \limsup_{t\to 0^+} \frac{\lambda(E\cap[x,x+t])}{t}$$

and

$$\overline{d}^{-}(E,x) = \liminf_{t\to 0^+} \frac{\lambda(E\cap[x-t,x])}{t}$$

are called the right upper density of E at x and left upper density of E at x, respectively. The number

$$\overline{d}(E,x) = \max\left\{\overline{d}^{+}(E,x), \overline{d}^{-}(E,x)\right\}$$

is called the upper density of E at x.

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Recall the definition of ϱ-upper continuous function.

Definition 1. [2] *Let E be a measurable subset of $\mathbb{R}$. If $x \in \mathbb{R}$ and $0 < \varrho < 1$, then we shall say that x is a point of ϱ-type upper density of E if $\overline{d}(E, x) > \varrho$.*

Definition 2. [2] *Let $x \in I$. A real-valued function f defined on I is called ϱ-upper continuous at x provided that there is a measurable set $E \subset I$ such that x is a point of ϱ-type upper density of E, $x \in E$ and $f|_E$ is continuous at x. If f is ϱ-upper continuous at each point of I, we say that f is ϱ-upper continuous.*

By $\mathcal{UC}_\varrho$ we denote the class of all ϱ-upper continuous functions defined in an open interval I.

2. Weakly ϱ-continuous functions

Now, we shall give the basic definitions of this paper.

Definition 3. *Let E be a measurable subset of $\mathbb{R}$ and $x \in \mathbb{R}$. If $\varrho \in (0, 1)$, then we say that x is a point of weak ϱ-type upper density of E if $\overline{d}(E, x) \geq \varrho$.*

Definition 4. *A real-valued function f defined in I is called weakly ϱ-upper continuous at $x \in I$ provided that there is a measurable set $E \subset I$ such that x is a point of weak ϱ-type upper density of E, $x \in E$ and $f|_E$ is continuous at x. If f is weakly ϱ-upper continuous at each point of I, we say that f is weakly ϱ-upper continuous.*

By $w\mathcal{UC}_\varrho$ we denote the class of all weakly ϱ-upper continuous functions defined on an open interval I.

In an obvious way we define one-sided weak ϱ-upper continuity at a point x and f is weakly ϱ-upper continuous at x if and only if it is weakly ϱ-upper continuous at x on the right or on the left.

Corollary 1. *If $0 < \varrho_1 < \varrho_2 < 1$, $x_0 \in I$ and $f : I \to \mathbb{R}$ is weakly ϱ_2-upper continuous at x_0, then f is weakly ϱ_1-upper continuous at x_0.*

Corollary 2. *If $0 < \varrho < 1$ and $f : I \to \mathbb{R}$ is ϱ-upper continuous at some point x_0 from I, then f is weakly ϱ-upper continuous at x_0.*

Example 1. *Let $\varrho \in (0, 1)$. We shall show that there exists $f : \mathbb{R} \to \mathbb{R}$ such that $f \in w\mathcal{UC}_\varrho \setminus \mathcal{UC}_\varrho$.*

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} x_n = 0$ and $x_{n+1} < x_n$ for every $n \geq 1$. For each $n \geq 1$ take any $y_n \in (x_{n+1}, x_n)$ such that $x_n - y_n = \varrho(x_n - x_{n+1})$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \{x_n\} \cup (x_1, \infty), \\ 1 & \text{if } x \in \{0\} \cup \bigcup_{n=1}^{\infty} [y_n, x_n), \\ \text{linear on each interval } [x_{n+1}, y_n], n \geq 1. \end{cases}$$

Clearly, f is ϱ-upper continuous at every point except at 0. Take any $\varepsilon > 0$. Then

$$\lambda\left(\{x \in [x_{n+1}, y_n] \colon |f(x) - 1| < \varepsilon\}\right) = \varepsilon\lambda\left([x_{n+1}, y_n]\right) = \\ = \varepsilon(1 - \varrho)\lambda\left([x_{n+1}, x_n]\right).$$

Therefore,

$$\lambda\left(\{x \in [0, x_n] \colon |f(x) - 1| < \varepsilon\}\right) = \\ = \sum_{k=n}^{\infty} \left(\varrho\lambda\left([x_{k+1}, x_k]\right) + \varepsilon(1 - \varrho)\lambda\left([x_{k+1}, x_k]\right)\right) = \\ = (\varrho + \varepsilon(1 - \varrho)) \sum_{k=n}^{\infty} \lambda\left([x_{k+1}, x_k]\right) = (\varrho + \varepsilon(1 - \varrho))\lambda\left([0, x_n]\right).$$

Then

$$\overline{d}\left(\{x \colon |f(x) - 1| < \varepsilon\}, 0\right) = \lim_{n\to\infty} \frac{\lambda\left(\{x \in [0, x_n] \colon |f(x) - 1| < \varepsilon\}\right)}{\lambda\left([0, x_n]\right)} = \\ = \varrho + \varepsilon(1 - \varrho).$$

Since $\lim_{\varepsilon\to 0^+} \overline{d}\left(\{x \colon |f(x) - 1| < \varepsilon\}, 0\right) = \lim_{\varepsilon\to 0^+} (\varrho + \varepsilon(1 - \varrho)) = \varrho$, we conclude that f is not ϱ-upper continuous at 0 and f is weakly ϱ-upper continuous at 0. Hence $f \in w\mathcal{UC}_\varrho \setminus \mathcal{UC}_\varrho$.

Corollary 3. *If $0 < \varrho_1 < \varrho_2 < 1$ and $f\colon I \to \mathbb{R}$ is weakly ϱ_2-upper continuous at some point x_0 from I, then f is ϱ_1-upper continuous at x_0.*

Example 2. *We shall show that if $0 < \varrho_1 < \varrho_2 < 1$, then there is a function $f\colon (a, b) \to \mathbb{R}$ such that $f \in \mathcal{UC}_{\varrho_1} \setminus w\mathcal{UC}_{\varrho_2}$.*

Let $a < 0 < b$. We can find a sequence $([a_n, b_n])_{n\geq 1}$ of pairwise disjoint closed intervals such that $0 < b_{n+1} < a_n < b_n$ for each n and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty} [a_n, b_n], 0\right) = \frac{\varrho_1 + \varrho_2}{2}.$$

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for every $n \geq 1$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty}([c_n, d_n] \setminus [a_n, b_n]), 0\right) = 0$. Put $I_n = [a_n, b_n]$, $J_n = [c_n, d_n]$ for every $n \geq 1$. Define a function $f\colon (a, b) \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 & \text{if } x \in \{0\} \cup \bigcup_{n=1}^{\infty} I_n, \\ 1 & \text{if } x \in (a, 0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] \cup [d_1, b), \\ \multicolumn{2}{l}{\text{linear on each interval } [c_n, a_n], [b_n, d_n], n \geq 1.} \end{cases}$$

The function f is continuous at every point except at 0. If $E = \bigcup_{n=1}^{\infty} I_n \cup \{0\}$, then the function f restricted to E is constant, so in particular, it is continuous at zero. Moreover,

$$\overline{d}(E, 0) \geq \overline{d}^+(E, 0) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) = \frac{\varrho_1 + \varrho_2}{2} > \varrho_1.$$

Hence $f \in \mathcal{UC}_{\varrho_1}$. But

$$\begin{aligned} \overline{d}^+(\{x\colon f(x) < 1\}, 0) &\leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} J_n, 0\right) \leq \\ &\leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) + \overline{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), 0\right) = \frac{\varrho_1 + \varrho_2}{2} < \varrho_2. \end{aligned}$$

Moreover $\overline{d}^-(\{x\colon f(x) < 1\}, 0) = 0$. Thus $\overline{d}(\{x\colon f(x) < 1\}, 0) < \varrho_2$ and f is not weakly ϱ_2-upper continuous at 0. Hence $f \notin w\mathcal{UC}_{\varrho_2}$.

Corollary 4. $\bigcup_{\varrho\in(0,1)} \mathcal{UC}_\varrho = \bigcup_{\varrho\in(0,1)} w\mathcal{UC}_\varrho$.

Corollary 5. $\bigcap_{\varrho\in(0,1)} \mathcal{UC}_\varrho = \bigcap_{\varrho\in(0,1)} w\mathcal{UC}_\varrho$.

Definition 5. *We say that a real-valued function f defined on an open interval I has Denjoy property at $x_0 \in I$ if for each $\varepsilon > 0$ and $\delta > 0$ the set*

$$\{x \in (x_0 - \delta, x_0 + \delta)\colon |f(x) - f(x_0)| < \varepsilon\}$$

contains a measurable subset of positive measure. We say that f has Denjoy property if it has Denjoy property at each point $x \in I$.

Immediately from Theorem 2.1 in [2], Remark 2.1 in [2] and Corollary 2 we obtain the following results.

Corollary 6. *If $0 < \varrho < 1$ and $f \in u\mathcal{UC}_\varrho$, then f is measurable.*

Corollary 7. *If $0 < \varrho < 1$ and $f \in u\mathcal{UC}_\varrho$, then f has Denjoy property.*

The proof of the next corollary follows directly from Theorem 2.4 in [4].

Corollary 8. *There exists function f such that $f \in \bigcap_{\varrho \in (0,1)} u\mathcal{UC}_\varrho$ and f does not belong to the Baire class 1.*

We shall need the following lemma.

Lemma 1. [3] *If $0 < \varrho \leq 1$ and $\{E_n : n \in \mathbb{N}\}$ is a descending family of measurable sets such that $x \in \bigcap_{n=1}^{\infty} E_n$ and $\overline{d}(E_n, x) \geq \varrho$ for $n \geq 1$, then there exists a measurable set E such that $\overline{d}(E, x) \geq \varrho$, $x \in E$, and for each $n \in \mathbb{N}$ there exists $\delta_n > 0$ for which $E \cap [x - \delta_n, x + \delta_n] \subset E_n$.*

We shall give an equivalent condition of weak ϱ-upper continuity at a point.

Theorem 1. *If $0 < \varrho < 1$ and $f : I \to \mathbb{R}$ is a measurable function, then f is weakly ϱ-upper continuous at $x \in I$ if and only if*

$$\overline{d}\,(\{y \in I : |f(x) - f(y)| < \varepsilon\}, x) \geq \varrho \quad \textit{for every} \quad \varepsilon > 0.$$

Proof. Assume that f is weakly ϱ-upper continuous at x. Let $E \subset I$ be a measurable set such that $x \in E$, $f|_E$ is continuous at x and $\overline{d}(E, x) \geq \varrho$. Since $f|_E$ is continuous at x, for each $\varepsilon > 0$ we can find $\delta > 0$ such that $[x - \delta, x + \delta] \cap E \subset \{y \in E : |f(x) - f(y)| < \varepsilon\}$. Hence for each $\varepsilon > 0$

$$\overline{d}\,(\{y \in I : |f(x) - f(y)| < \varepsilon\}, x) \geq \overline{d}\,(\{y \in E : |f(x) - f(y)| < \varepsilon\}, x) = \\ = \overline{d}(E, x) \geq \varrho.$$

Finally, assume that for each $\varepsilon > 0$,

$$\overline{d}\big(\{y \in I : |f(x) - f(y)| < \varepsilon\}, x\big) \geq \varrho.$$

By Lemma 1 for sets $E_n = \big\{y \in I : |f(x) - f(y)| < \frac{1}{n}\big\}$, where $n \in \mathbb{N}$, we can construct a measurable set E such that $x \in E$, $\overline{d}(E, x) \geq \varrho$ and for each n there exists $\delta_n > 0$ for which $E \cap [x - \delta_n, x + \delta_n] \subset E_n$. The last condition implies that $f|_E$ is continuous at x. It follows that f is weakly ϱ-upper continuous at x, what was to be shown. $\square$

Now we will show that the family of weakly ϱ-upper continuous functions is closed under uniform limits, i.e. every limit of uniformly convergent sequence of functions from $u\mathcal{UC}_\varrho$ belongs to this family.

Theorem 2. *If $0 < \varrho < 1$ and a sequence $(f_n)_{n\geq 1}$ of weakly ϱ-upper continuous functions is uniformly convergent to a function f, then f is weakly ϱ-upper continuous.*

Proof. Let $(f_n)_{n\geq 1}$ be a sequence of weakly ϱ-upper continuous functions uniformly converges to f. Let $x_0 \in I$ and $\varepsilon > 0$. There exists $n_0 \geq 1$ such that for every $k > n_0$ and every $x \in I$ the inequality

$$|f_k(x) - f(x)| < \frac{\varepsilon}{3}$$

holds. Fix $n > n_0$. Since f_n is weakly ϱ-upper continuous at x_0, there exists a measurable set $E \subset I$ such that $x_0 \in E$, $f_n|_E$ is continuous at x_0 and $\overline{d}(E, x_0) \geq \varrho$. Then there exists a positive δ such that

$$[x_0 - \delta, x_0 + \delta] \cap E \subset \left\{x \in E \colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\}.$$

Notice that

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

if $x \in [x_0 - \delta, x_0 + \delta] \cap E$. Therefore

$$\left\{x \in E \colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\} \subset \{x \colon |f(x) - f(x_0)| < \varepsilon\}.$$

Hence

$$\begin{aligned}\overline{d}\left(\{x \colon |f(x) - f(x_0)| < \varepsilon\}, x_0\right) &\geq \\ &\geq \overline{d}\left(\left\{x \in E \colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\}, x_0\right) = \overline{d}(E, x_0).\end{aligned}$$

Therefore

$$\overline{d}\left(\{x \colon |f(x) - f(x_0)| < \varepsilon\}, x_0\right) \geq \overline{d}(E, x_0) \geq \varrho.$$

It means that the function f is weakly ϱ-upper continuous at x_0. □

Example 3. *We shall show that the family of ϱ-upper continuous functions is not closed under the operation of uniform convergence.*

Define the function f in the same way as in Example 1. Let $f_n \colon \mathbb{R} \to \mathbb{R}$, $f_n = \min\{1 - \frac{1}{n}, f\}$ for each $n \geq 1$. Then, clearly, the sequence $(f_n)_{n\geq 1}$ uniformly converges to f and $f \notin \mathcal{UC}_\varrho$. Since

$$\left\{x \colon f_n(x) = 1 - \tfrac{1}{n} = f(0)\right\} = \left\{x \colon |f(x) - 1| < \tfrac{1}{n}\right\}$$

and $\overline{d}\left(\{x \colon |f(x) - 1| < \frac{1}{n}\}, 0\right) = \varrho + \frac{1}{n}(1 - \varrho) > \varrho$, we infer that $f_n \in \mathcal{UC}_\varrho$ for each $n \geq 1$.

3. Maximal additive family

Definition 6. *Let $\mathcal{F}$ be any family of real valued functions defined on I. The set $\mathcal{M}_a(\mathcal{F}) = \{g \colon \forall_{f\in\mathcal{F}}\ f+g \in \mathcal{F}\}$ is called a maximal additive family for $\mathcal{F}$.*

Remark 1. *If a zero (constant) function is a member of a family of functions F, then $\mathcal{M}_a(\mathcal{F}) \subset \ \mathcal{F}$.*

Lemma 2. [3] *Let numbers c and γ fulfil the inequality $0 < c < \gamma < 1$. Moreover, let E be a measurable subset of $\mathbb{R}$ with the property $\overline{d}^{+}(E, x) = c$ for some point $x \in \mathbb{R}$. Then there exists a measurable set H such that $E \subset H$, $\overline{d}^{+}(H, x) \geq \gamma$ and $\overline{d}^{+}(H \setminus E, x) \leq \gamma - c(1-\gamma)$.*

The proof of next theorem is based on the proof of Theorem 2.1 in [3], where the maximal additive class for ϱ-upper continuous functions is discussed.

Theorem 3. *If $0 < \varrho < 1$, then for each $f \in u\mathcal{UC}_{\varrho} \setminus \mathcal{A}$ there exists $g \colon I \to \mathbb{R}$ such that $g \in u\mathcal{UC}_{\varrho}$ and $f + g \notin u\mathcal{UC}_{\varrho}$.*

Proof. Since $f \notin \mathcal{A}$, there exist $x_0 \in I$ and $\varepsilon > 0$ such that

$$\overline{d}^{+}\left(\{x \in I \colon |f(x) - f(x_0)| \geq \varepsilon\}, x_0\right) > 0$$

or

$$\overline{d}^{-}\left(\{x \in I \colon |f(x_0) - f(x)| \geq \varepsilon\}, x_0\right) > 0.$$

Without loss of generality we may assume that the first inequality holds.

Put $E = \{x \in I \colon |f(x) - f(x_0)| \geq \varepsilon\}$ and $c = \overline{d}^{+}(E, x_0)$. Therefore $c > 0$. Let γ be a real number satisfying conditions $\gamma \geq \varrho$, $c < \gamma < 1$ and $\gamma - c(1-\gamma) < \varrho$. By Lemma 2, there exists a measurable set H such that $E \subset H$, $\overline{d}^{+}(H, x_0) \geq \gamma$ and $\overline{d}^{+}(H \setminus E, x_0) \leq \gamma - c(1-\gamma)$. Next one can find a sequence $([a_n, b_n])_{n\geq 1}$ of closed intervals such that $x_0 < b_{n+1} < a_n < b_n$ for each $n \geq 1$ and

$$\overline{d}^{+}\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \setminus H, x_0\right) = \overline{d}^{+}\left(H \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], x_0\right) = 0.$$

Thus $\overline{d}^{+}\left(\bigcup_{n=1}^{\infty} [a_n, b_n], x_0\right) = \overline{d}^{+}(H, x_0) \geq \gamma \geq \varrho$.

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for all n and $\overline{d}^{+}\left(\bigcup_{n=1}^{\infty} ([c_n, d_n] \setminus [a_n, b_n]), x_0\right) = 0$. Put $I_n = [a_n, b_n]$ and $K_n = [c_n, d_n]$ for each $n \geq 1$. Define a function $g \colon (a, b) \to \mathbb{R}$ letting

$$g(x) = \begin{cases} 0 & \text{if } x \in \{x_0\} \cup \bigcup\limits_{n=1}^{\infty} I_n, \\ -f(x) + f(x_0) + \varepsilon & \text{if } x \in (a, x_0) \cup \bigcup\limits_{n=1}^{\infty} [d_{n+1}, c_n] \cup [d_1, b), \\ \text{linear on each interval } [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

Since $f \in u\mathcal{UC}_\varrho$, g is weakly ϱ-upper continuous at each point except at x_0. Applying inequality

$$\overline{d}\left(\{x \colon g(x) = g(x_0) = 0\}, x_0\right) \geq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) \geq \varrho,$$

we conclude that g is weakly ϱ-upper continuous at x_0, too. It means that $g \in u\mathcal{UC}_\varrho$.

Now, we shall show that $f + g$ is not weakly ϱ-upper continuous at x_0. Put

$$F = \{x \in I \colon \ |(f+g)(x) - (f+g)(x_0)| < \varepsilon\}.$$

For $x \notin \bigcup\limits_{n=1}^{\infty} K_n \cup \{x_0\}$, we have $(f+g)(x) - (f+g)(x_0) = \varepsilon$. Therefore $F \subset \bigcup\limits_{n=1}^{\infty} K_n \cup \{x_0\}$. If $x \in \bigcup\limits_{n=1}^{\infty} I_n$, then $(f+g)(x) - (f+g)(x_0) = f(x) - f(x_0)$ and consequently $\bigcup\limits_{n=1}^{\infty} I_n \cap F \subset \bigcup\limits_{n=1}^{\infty} I_n \setminus E$. Thus

$$\begin{aligned} \overline{d}(F, x_0) = \overline{d}^+(F, x_0) \leq \overline{d}^+\left(F \cap \bigcup_{n=1}^{\infty} I_n, x_0\right) + \overline{d}^+\left(F \setminus \bigcup_{n=1}^{\infty} I_n, x_0\right) \leq \\ \leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \setminus E, x_0\right) + \overline{d}^+\left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n), x_0\right) = \\ = \overline{d}^+(H \setminus E, x_0) \leq \gamma - c(1-\gamma) < \varrho. \end{aligned}$$

It follows that $f + g$ is not weakly ϱ-upper continuous at x_0. Hence $f + g \notin u\mathcal{UC}_\varrho$, which completes the proof. $\square$

The proof of the following lemma is identical to the proof of Lemma 2.2 in [3] and we omit it.

Lemma 3. *Let $f \colon I \to \mathbb{R}$, $g \colon I \to \mathbb{R}$ be weakly ϱ-upper continuous at some point $x \in I$, where $0 < \varrho < 1$. If at least one of those functions is approximately continuous at x, then $f + g$ and $f \cdot g$ are weakly ϱ-upper continuous at x.*

Corollary 9. *Let $g\colon I \to \mathbb{R}$ be weakly ϱ-upper continuous at some point $x \in I$, where $0 < \varrho < 1$. If $f\colon I \to \mathbb{R}$ is approximately continuous at x, then $f + g$ and $f \cdot g$ are weakly ϱ-upper continuous at x.*

Corollary 10. *Let $f\colon I \to \mathbb{R}$, $g\colon I \to \mathbb{R}$ be weakly ϱ-upper continuous in I, where $0 < \varrho < 1$. If $D_{ap}(f) \cap D_{ap}(g) = \emptyset$, where $D_{ap}(f)$ denotes the set of all points at which f is not approximately continuous, then $f + g$ and $f \cdot g$ are weakly ϱ-upper continuous in I.*

Theorem 4. *If $0 < \varrho < 1$, then $\mathcal{M}_a(w\mathcal{UC}_\varrho) = \mathcal{A}$.*

Proof. By Theorem 3, we have $w\mathcal{UC}_\varrho \cap \mathcal{M}_a(w\mathcal{UC}_\varrho) \subset \mathcal{A}$. By Remark 1, we have the inclusion $\mathcal{M}_a(w\mathcal{UC}_\varrho) \subset w\mathcal{UC}_\varrho$. Therefore $\mathcal{M}_a(w\mathcal{UC}_\varrho) \subset \mathcal{A}$. Finally, by Lemma 9, we have $\mathcal{A} \subset \mathcal{M}_a(w\mathcal{UC}_\varrho)$. □

4. Maximal multiplicative family

Definition 7. *If $\mathcal{F}$ is any family of real valued functions defined on an open interval I, then the set $\{g\colon \forall_{f\in\mathcal{F}}\ f\cdot g \in \mathcal{F}\}$ is called a maximal multiplicative family for $\mathcal{F}$ and is denoted by $\mathcal{M}_m(\mathcal{F})$.*

Remark 2. *If a constant function equalled to 1 is a member of a family of functions $\mathcal{F}$, then $\mathcal{M}_m(\mathcal{F}) \subset \mathcal{F}$.*

Lemma 4. *If $0 < \varrho < 1$ and a measurable function $f\colon I \to \mathbb{R}$ is not approximately continuous at some point x_0 from I for which $f(x_0) \neq 0$, then there exists $g\colon I \to \mathbb{R}$ such that $g \in w\mathcal{UC}_\varrho$ and $f \cdot g \notin w\mathcal{UC}_\varrho$.*

Proof. Without loss of generality we may assume that f is not approximately continuous from right side at x_0. Then we can find a positive ε such that $\varepsilon < |f(x_0)|$ and

$$\overline{d}^+\big(\{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}, x_0\big) = c > 0.$$

Put $E = \{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}$. Take γ such that

$$\varrho \leq \gamma < 1, \quad c < \gamma \quad \text{and} \quad \gamma - c(1-\gamma) < \varrho.$$

By Lemma 2, there exists a measurable set H such that

$$E \subset H, \quad \overline{d}^+(H, x_0) \geq \gamma \quad \text{and} \quad \overline{d}^+(H \setminus E, x_0) \leq \gamma - c(1-\gamma).$$

Similarly as in proof of Lemma 3.1 in [3] we can find a sequence $([a_n, b_n])_{n\geq 1}$ of closed intervals such that $x_0 < b_{n+1} < a_n < b_n$ for each $n \geq 1$ and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n] \setminus H, x_0\right) = \overline{d}^+\left(H \setminus \bigcup_{n=1}^{\infty}[a_n, b_n], x_0\right) = 0.$$

Then $\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], x_0\right) = \overline{d}^+(H, x_0) \geq \gamma \geq \varrho$.

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for all n and $\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}([c_n, d_n] \setminus [a_n, b_n]), x_0\right) = 0$. Denote now $I_n = [a_n, b_n]$, $K_n = [c_n, d_n]$ for each $n \geq 1$. Define a function $g\colon (a, b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} I_n \cup [b_1, b), \\ 0 & \text{if } x \in (a, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \multicolumn{2}{l}{\text{linear on each interval } [c_n, a_n], [b_{n+1}, d_{n+1}], n \geq 1.} \end{cases}$$

Then g is continuous except at x_0. Moreover,

$$\overline{d}\left(\{x \in I\colon g(x) = 1\}, x_0\right) \geq \overline{d}^{+}\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) = \overline{d}^{+}(H, x_0) \geq \gamma \geq \varrho$$

and g restricted to $\{x \in I\colon g(x) = 1\}$ is continuous at x_0. It follows that g is weakly ϱ-upper continuous at x_0. Therefore $g \in w\mathcal{UC}_{\varrho}$.

Moreover, $(f \cdot g)(x_0) = f(x_0)$ and

$$\{x\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\} \cap \left((a, x_0) \cup \bigcup_{n=1}^{\infty}[d_{n+1}, c_n]\right) = \emptyset.$$

Then

$$\begin{aligned} &\overline{d}\left(\{x \in I\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\}, x_0\right) \leq \\ &\leq \overline{d}^{+}\left(\left\{x \in \bigcup_{n=1}^{\infty} K_n\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\right\}, x_0\right) = \\ &= \overline{d}^{+}\left(\left\{x \in \bigcup_{n=1}^{\infty} I_n\colon\ |f(x) - f(x_0)| < \varepsilon\right\}, x_0\right) = \\ &= \overline{d}^{+}\left(\{x \in H\colon\ |f(x) - f(x_0)| < \varepsilon\}, x_0\right) = \\ &= \overline{d}^{+}\left(\{x \in H \setminus E\colon\ |f(x) - f(x_0)| < \varepsilon\}, x_0\right) \leq \gamma - c(1-\gamma) < \varrho. \end{aligned}$$

It implies that $f \cdot g$ is not weakly ϱ-upper continuous at x_0 i.e. $fg \notin w\mathcal{UC}_{\varrho}$. $\square$

Definition 8. *If $0 < \varrho < 1$, then by $\mathcal{W}(\varrho)$ we shall denote the family of all measurable functions $f\colon I \to \mathbb{R}$ such that at each $x_0 \in D_{ap}(f)$ the following two conditions hold*

(W1) $f(x_0) = 0$ (*in other words $D_{ap}(f) \subset N_f$, where $N_f = \{x\colon f(x) = 0\}$*);

$(W2)$ *for each* $\varepsilon > 0$ *and for each measurable set* F *such that* $F \supset N_f$ *and* $\overline{d}(F, x_0) \geq \varrho$ *we have*

$$\overline{d}(F \cap \{x \in I \colon |f(x) - f(x_0)| < \varepsilon\}, x_0) \geq \varrho.$$

Theorem 5. $\mathcal{M}_m(u\mathcal{UC}_\varrho) = \mathcal{W}(\varrho)$ *for each* ϱ *such that* $0 < \varrho < 1$.

Proof. Fix ϱ from the interval $(0,1)$. Let $f \in \mathcal{W}(\varrho)$ and $g \in u\mathcal{UC}_\varrho$. Take any $x_0 \in I$. Then we can find a measurable set E such that $x_0 \in E$, $\overline{d}(E, x_0) \geq \varrho$ and $g|_E$ is continuous at x_0.

First, we assume that f is approximately continuous at x_0. Then, by Lemma 9, $f \cdot g$ is weakly ϱ-upper continuous at x_0.

Now, we assume that $x_0 \in D_{ap}(f)$. By condition $(W1)$, we obtain $f(x_0) = 0$. Since $g|_E$ is continuous at x_0, there exist positive numbers r and M such that $|g(x)| < M$ for $x \in E \cap [x_0 - r, x_0 + r]$. Put $F = E \cup N_f$. Then $N_f \subset F$ and $\overline{d}(F, x_0) \geq \varrho$. Let $\varepsilon > 0$. Then

$$\begin{aligned}\{x \in I \colon |(f \cdot g)(x)| < \varepsilon\} \cap [x_0 - r, x_0 + r] \supset& \\ \supset F \cap \left\{x \in I \colon |f(x)| < \tfrac{\varepsilon}{M}\right\} &\cap [x_0 - r, x_0 + r].\end{aligned}$$

By condition $(W2)$, we have

$$\begin{aligned}\overline{d}\big(\{x \colon |(f \cdot g)(x)| < \varepsilon\}, x_0\big) \geq \overline{d}\left(\left\{x \colon |f(x)| < \tfrac{\varepsilon}{M}\right\} \cap F, x_0\right) =& \\ = \overline{d}\big(\{x \colon |f(x)| < \varepsilon'\} \cap F, x_0\big) &\geq \varrho,\end{aligned}$$

where $\varepsilon' = \frac{\varepsilon}{M}$. By Theorem 1, $f \cdot g$ is weakly ϱ-upper continuous at x_0. Hence $f \cdot g \in u\mathcal{UC}_\varrho$. In this way, we have proven that $\mathcal{W}(\varrho) \subset \mathcal{M}_m(u\mathcal{UC}_\varrho)$.

Finally, assume that $f \in \mathcal{M}_m(u\mathcal{UC}_\varrho)$. If $x_0 \in D_{ap}(f)$, then, by Lemma 4, we obtain $f(x_0) = 0$. Therefore f satisfies the condition $(W1)$. Take any measurable set F such that $N_f \subset F$ and $\overline{d}(F, x) \geq \varrho$. Identically as in the proof of Theorem 3.1 in [3] we can find sequences $([a_n, b_n])_{n\geq1}$, $([c_n, d_n])_{n\geq1}$, $([a'_n, b'_n])_{n\geq1}$, $([c'_n, d'_n])_{n\geq1}$, $(\alpha_n)_{n\geq1}$, $(\alpha'_n)_{n\geq1}$ that satisfy conditions listed in that proof.

Define a function $g \colon (a,b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & \text{if } x \in \bigcup\limits_{n=1}^{\infty} [a_n, b_n] \cup \bigcup\limits_{n=1}^{\infty} [a'_n, b'_n] \cup (a, a'_1] \cup [b_1, b) \cup \{x_0\}, \\ \alpha_n, & \text{if } x \in [d_{n+1}, c_n],\ n = 1, 2, \ldots, \\ \alpha'_n, & \text{if } x \in [d'_n, c'_{n+1}],\ n = 1, 2, \ldots, \\ \multicolumn{2}{l}{\text{linear on each } [c_n, a_n], [b_{n+1}, d_{n+1}], [c'_{n+1}, a'_{n+1}], [b'_n, d'_n], n \geq 1.} \end{cases}$$

It follows directly from the definition of g, that g is continuous at each point except at x_0. Since

$$\overline{d}\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [a'_n, b'_n], x_0\right) = \overline{d}(F, x_0)$$

and g restricted to the set $\bigcup_{n=1}^{\infty}[a_n, b_n] \cup \bigcup_{n=1}^{\infty}[a'_n, b'_n] \cup \{x_0\}$ is constant, g is weakly ϱ-upper continuous at x_0. Thus $g \in u\mathcal{UC}_\varrho$. Hence $f \cdot g \in u\mathcal{UC}_\varrho$. Moreover, $(f \cdot g)(x_0) = 0$. Put

$$E_\varepsilon = \{x \in I \colon |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\} = \{x \in I \colon |(f \cdot g)(x)| < \varepsilon\}$$

if $0 < \varepsilon < 1$. Since $f \cdot g \in u\mathcal{UC}_\varrho$, $\overline{d}(E_\varepsilon, x_0) \geq \varrho$. On the other hand, in the same way as in mentioned proof, we obtain

$$\overline{d}^+(E_\varepsilon, x_0) \leq \overline{d}^+\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big),$$

$$\overline{d}^-(E_\varepsilon, x_0) \leq \overline{d}^-\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big)$$

if $0 < \varepsilon < 1$. Thus $\overline{d}\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big) \geq \overline{d}(E_\varepsilon, x_0) \geq \varrho$. It follows that the condition $(W2)$ is satisfied and $f \in \mathcal{W}(\varrho)$. $\square$

Corollary 11. *If a measurable function $f \colon I \to \mathbb{R}$ satisfies the following conditions:*

(1) $x_0 \in D_{ap}(f)$,
(2) $\overline{d}(N_f, x_0) \geq \varrho$,
(3) $f(x_0) = 0$

for some $x_0 \in I$ and $\varrho \in (0, 1)$, then $f \in \mathcal{W}(\varrho)$.

Corollary 12. $\mathcal{M}_a(u\mathcal{UC}_\varrho) = \mathcal{A} \subsetneq \mathcal{M}_m(u\mathcal{UC}_\varrho)$.

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AXISYMMETRIC THERMAL STRESSES IN A HALF-SPACE IN THE FRAMEWORK OF FRACTIONAL THERMOELASTICITY

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Abstract

A theory of thermal stresses based on the time-fractional heat conduction equation is considered. The Caputo fractional derivative is used. The fundamental solution to the axisymmetric heat conduction equation in a half-space under the Dirichlet boundary condition and the associated thermal stresses are investigated.

1. Introduction

Numerical applications of fractional calculus to problems of mechanics can be found in the literature. We can quote investigations on viscoelasticity [6], creep [20], hereditary mechanics of solids [21], Brownian motion [5], stress and strain localization in solids [1] (see also [4], [22], [23], [26], [27]). The theory of thermal stresses based on the time-fractional heat conduction equation was proposed by the author [11] and was developed in the subsequent studies [12], [13], [15]–[17]. Axisymmetric problems for the time-fractional heat conduction equation in a half-space were investigated in [14]. In this paper we study associated thermal stresses.

2. Formulation of the problem

A thermoelastic state of a solid is governed by the equilibrium equation in terms of displacements

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\mathbf{u} = \beta_T K_T\,\mathrm{grad}\,T, \tag{1}$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_T K_T T)\mathbf{I}, \tag{2}$$

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and the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \quad 0 < \alpha \le 2. \tag{3}$$

Here $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}$ the linear strain tensor, T the temperature, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T is the thermal coefficient of volumetric expansion, a denotes the thermal diffusivity, $\mathbf{I}$ stands for the unit tensor, and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative [2], [3], [10]

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n,$$

where $\Gamma(x)$ is the gamma function.

The Caputo derivative has the following Laplace transform rule

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n,$$

where the asterisk denotes the Laplace transform, s is the transform variable.

In this paper we will consider the axisymmetric fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \quad 0 < \alpha \le 2, \tag{4}$$

in the domain $0 \le r < \infty$, $0 < z < \infty$, $0 < t < \infty$ under zero initial conditions

$$t = 0: \qquad T = 0, \qquad 0 < \alpha \le 2, \tag{5}$$

$$t = 0: \qquad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6}$$

and the Dirichlet boundary condition

$$z = 0: \quad T = f(r,t). \tag{7}$$

The zero conditions at infinity

$$\lim_{r\to\infty} T(r,z,t) = 0, \quad \lim_{z\to\infty} T(r,z,t) = 0 \tag{8}$$

are also assumed.

In the quasi-static statement of the thermoelasticity problem, initial values of mechanical quantities are not considered. The boundary of a half-space is load free, hence

$$z = 0: \quad \sigma_{zz} = 0, \quad \sigma_{rz} = 0. \tag{9}$$

3. Representation of stresses

Just as in the classical theory of thermal stresses [8], [9], we can introduce the displacement potential Φ

$$\mathbf{u} = \operatorname{grad}\Phi. \tag{10}$$

In the quasi-static case, from the equilibrium equation (1) we get

$$\Delta\Phi = mT, \quad m = \frac{1+\nu}{1-\nu}\,\frac{\beta_T}{3}, \tag{11}$$

with ν being the Poisson ratio. The part of stresses due to the displacement potential Φ describes the influence of the temperature field and is given as

$$\boldsymbol{\sigma}^{(1)} = 2\mu\left(\operatorname{grad}\operatorname{grad}\Phi - \mathbf{I}\,\Delta\Phi\right). \tag{12}$$

In cylindrical coordinates in the case of axial symmetry

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = mT, \tag{13}$$

$$\sigma_{rr}^{(1)} = 2\mu\left[\frac{\partial^2\Phi}{\partial r^2} - \Delta\Phi\right], \tag{14}$$

$$\sigma_{\theta\theta}^{(1)} = 2\mu\left[\frac{1}{r}\,\frac{\partial\Phi}{\partial r} - \Delta\Phi\right], \tag{15}$$

$$\sigma_{zz}^{(1)} = 2\mu\left[\frac{\partial^2\Phi}{\partial z^2} - \Delta\Phi\right], \tag{16}$$

$$\sigma_{rz}^{(1)} = 2\mu\,\frac{\partial^2\Phi}{\partial r\partial z}. \tag{17}$$

The Hankel transform of order n with respect to the radial coordinate r

$$\mathcal{H}_{(n)}\left\{f(r)\right\} = \int_0^\infty f(r)\,J_n(r\xi)\,r\,\mathrm{d}r,$$

$$f(r) = \int_0^\infty \mathcal{H}_{(n)}\left\{f(r)\right\}\,J_n(r\xi)\,\xi\,\mathrm{d}\xi$$

is often used for solving problems in cylindrical coordinates. The following formulae are helpful in applications [25]

$$\mathcal{H}_{(n)}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{n^2}{r^2}\,f(r)\right\} = -\xi^2\mathcal{H}_{(n)}\left\{f(r)\right\},$$

$$\mathcal{H}_{(1)}\left\{\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi\,\mathcal{H}_{(0)}\left\{f(r)\right\},$$

$$\mathcal{H}_{(2)}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} - \frac{1}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = \xi^2\mathcal{H}_{(0)}\left\{f(r)\right\}.$$

In the case $n = 0$, simultaneously with the notation $\mathcal{H}_{(0)}\left\{f(r)\right\}$, we will use the notation $\mathcal{H}_{(0)}\left\{f(r)\right\} = \widehat{f}(\xi)$.

From (14)–(17) we have

$$\mathcal{H}_{(0)}\left\{\sigma_{rr}^{(1)} + \sigma_{\theta\theta}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi} - 4\mu\frac{\partial^2\widehat{\Phi}}{\partial z^2}, \tag{18}$$

$$\mathcal{H}_{(2)}\left\{\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi}, \tag{19}$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi}, \tag{20}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(1)}\right\} = -2\mu\,\xi\frac{\partial\widehat{\Phi}}{\partial z}. \tag{21}$$

It follows from (11) that

$$\frac{\partial^2\widehat{\Phi}}{\partial z^2} - \xi^2\widehat{\Phi} = m\,\widehat{T}. \tag{22}$$

The general solution of the homogeneous equation (22) has the form

$$\widehat{\Phi} = C_1\mathrm{e}^{-\xi z} + C_2\mathrm{e}^{\xi z}, \tag{23}$$

where the integration constant C_2 should be equal to zero according to the condition at infinity (8). To find the particular solution of the non-homogeneous equation (22) we consider the following equation

$$\frac{\partial^2\widehat{\Phi}}{\partial z^2} - \xi^2\widehat{\Phi} = \delta(z), \tag{24}$$

where $\delta(z)$ is the Dirac delta function. The solution of (24) is written as

$$\widehat{\Phi} = -\frac{1}{2\xi}\mathrm{e}^{-\xi|z|}. \tag{25}$$

Hence, the particular solution of (22) is represented in the convolution form

$$\widehat{\Phi}(\xi, z, t) = -\frac{1}{2\xi}\int_0^\infty \widehat{T}(\xi, \eta, t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{d}\eta. \tag{26}$$

Assuming in (23) $C_1 = 0$, we get

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(1)}\right\} = -\mu m\xi\int_0^\infty \widehat{T}(\xi, \eta, t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{d}\eta, \tag{27}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(1)}\right\} = -\mu m\xi\int_0^\infty \widehat{T}(\xi, \eta, t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{sign}\,(z-\eta)\,\mathrm{d}\eta. \tag{28}$$

The part of stress field expressed in terms of the biharmonic Love function

$$\sigma_{rr}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{\partial^2 L}{\partial r^2}\right], \tag{29}$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{1}{r}\frac{\partial L}{\partial r}\right], \tag{30}$$

$$\sigma_{zz}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[(2-\nu) \Delta L - \frac{\partial^2 L}{\partial z^2}\right], \tag{31}$$

$$\sigma_{rz}^{(2)} = 2\mu \frac{\partial}{\partial r}\left[(1-\nu) \Delta L - \frac{\partial^2 L}{\partial z^2}\right] \tag{32}$$

with

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 L = 0 \tag{33}$$

allows us to satisfy the prescribed boundary conditions for the components of the total stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}$.

In the Hankel transform domain the biharmonic equation for the Love function

$$\left(\frac{\partial^2}{\partial z^2} - \xi^2\right)^2 \widehat{L} = 0 \tag{34}$$

has the solution bounded at $z \to \infty$:

$$\widetilde{L} = (A + B\xi z)\, \mathrm{e}^{-\xi z}, \tag{35}$$

where A and B are constants which should be found from the boundary conditions.

In the Hankel transform domain we have

$$\mathcal{H}_{(0)}\left\{\sigma_{rr}^{(2)} + \sigma_{\theta\theta}^{(2)}\right\} = 2\mu\xi^3 \left[-A + (1+4\nu)B - B\xi z\right] \mathrm{e}^{-\xi z}, \tag{36}$$

$$\mathcal{H}_{(2)}\left\{\sigma_{rr}^{(2)} - \sigma_{\theta\theta}^{(2)}\right\} = 2\mu\xi^3 \left(A - B + B\xi z\right) \mathrm{e}^{-\xi z}, \tag{37}$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(2)}\right\} = 2\mu\xi^3 \left[A + (1-2\nu)B + B\xi z\right] \mathrm{e}^{-\xi z}, \tag{38}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(2)}\right\} = 2\mu\xi^3 \left(A - 2\nu B + B\xi z\right) \mathrm{e}^{-\xi z}. \tag{39}$$

From the load free boundary condition (9) we obtain

$$\mathcal{H}_0\left\{\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)}\right\} = 0, \quad \mathcal{H}_1\left\{\sigma_{rz}^{(1)} + \sigma_{rz}^{(2)}\right\} = 0, \tag{40}$$

$$A = -\frac{(1-4\nu)m}{2\xi^2}\int_0^\infty \widetilde{T}(\xi,\eta,t)\, \mathrm{e}^{-\xi\eta}\, \mathrm{d}\eta, \qquad B = \frac{m}{\xi^2}\int_0^\infty \widetilde{T}(\xi,\eta,t)\, \mathrm{e}^{-\xi\eta}\, \mathrm{d}\eta,$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}\right\} = \mu m \int_0^\infty \xi\, \widehat{T}(\xi,\eta,t) \left[(2\xi z+1)\,\mathrm{e}^{-\xi(z+\eta)} - \mathrm{e}^{-\xi|z-\eta|}\right] \mathrm{d}\eta,$$

$$\mathcal{H}_{(1)}\{\sigma_{rz}\} = \mu m \int_0^\infty \xi \widehat{T}(\xi,\eta,t)\Big[(2\xi z-1)\,\mathrm{e}^{-\xi(z+\eta)} - \mathrm{e}^{-\xi|z-\eta|}\,\mathrm{sign}\,(z-\eta)\Big]\,\mathrm{d}\eta.$$

4. Solution to the fractional heat conduction equation

We start from the fundamental solution to the Dirichlet problem for the time-fractional heat conduction equation (4) with zero initial conditions (5) and (6) and the prescribed boundary value of temperature

$$z = 0: \quad T = \frac{U_0}{2\pi r}\,\delta(r)\,\delta(t). \tag{41}$$

In the case of the Dirichlet boundary condition at a surface $z = 0$ the sin-Fourier transform is used:

$$\mathcal{F}\left\{f(z)\right\} = \widetilde{f}(\eta) = \int_0^\infty f(z)\,\sin(z\eta)\,\mathrm{d}z,$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\eta)\right\} = f(z) = \frac{2}{\pi}\int_0^\infty \widetilde{f}(\eta)\,\sin(z\eta)\,\mathrm{d}\eta,$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2}\right\} = -\eta^2\widetilde{f}(\eta) + \eta f(z)\Big|_{z=0},$$

where the tilde denotes the Fourier transform, η is the transform variable.

Applying the Laplace transform with respect to time t, the Hankel transform with respect to the radial coordinate r, and the sin-Fourier transform with respect to the spatial coordinate z gives

$$\widetilde{\widehat{T}}^{*} = \frac{aU_0\eta}{2\pi}\,\frac{1}{s^\alpha + a(\xi^2+\eta^2)} \tag{42}$$

or after inversion of integral transforms [14]

$$T = \frac{aU_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] J_0(r\xi)\,\sin(z\eta)\,\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{43}$$

Here the following formula [10]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + b}\right\} = t^{\beta-1}\,E_{\alpha,\beta}(-bt^\alpha)$$

has been used, where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function in two parameters α and β [3].

It should be emphasized that partition of the total stress tensor $\boldsymbol{\sigma}$ into the stress tensors $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ is not unique. Sometimes, it is helpful to suppose that $\Phi\big|_{z=0} = 0$ [8]. In this case, the sin-Fourier transform with respect to the spatial coordinate z (as well as the Laplace transform with respect to time t) can be applied to (22) resulting in

$$\begin{aligned}\Phi = &-\frac{aU_0 m t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]\\ &\times\frac{\xi\,\eta}{\xi^2+\eta^2}\,J_0(r\xi)\,\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{44}$$

The stress tensor $\boldsymbol{\sigma}^{(1)}$ has the following components:

$$\begin{aligned}\sigma_{rr}^{(1)} = &-\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]\\ &\times\frac{\xi\,\eta}{\xi^2+\eta^2}\left[\frac{\xi}{r}\,J_1(r\xi)+\eta^2 J_0(r\xi)\right]\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned} \tag{45}$$

$$\begin{aligned}\sigma_{\theta\theta}^{(1)} = &-\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]\\ &\times\frac{\xi\,\eta}{\xi^2+\eta^2}\left[\left(\xi^2+\eta^2\right)J_0(r\xi)-\frac{\xi}{r}\,J_1(r\xi)\right]\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned} \tag{46}$$

$$\begin{aligned}\sigma_{zz}^{(1)} = &-\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]\\ &\times\frac{\xi^3\eta}{\xi^2+\eta^2}\,J_0(r\xi)\,\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned} \tag{47}$$

$$\begin{aligned}\sigma_{rz}^{(1)} = &\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]\\ &\times\frac{\xi^2\,\eta^2}{\xi^2+\eta^2}\,J_0(r\xi)\,\cos(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{48}$$

It follows from the load free condition (41) that

$$A = -(1-2\nu)B, \tag{49}$$

$$B = \frac{m a U_0 t^{\alpha-1}}{\pi^2}\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right]\frac{\eta^2}{\xi^2(\xi^2+\eta^2)}\,\mathrm{d}\eta. \tag{50}$$

To investigate several particular cases of the obtained solution it is convenient to pass to polar coordinates in the (ξ,η)-domain:

$$\xi = \varrho\cos\vartheta, \qquad \eta = \varrho\sin\vartheta.$$

Equation (43) for temperature is rewritten as

$$T = \frac{aU_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \varrho^3 E_{\alpha,\alpha}(-a\varrho^2 t^\alpha)\,\mathrm{d}\varrho \times \int_0^{\pi/2} J_0(r\varrho\cos\vartheta)\,\sin(z\varrho\sin\vartheta)\,\sin\vartheta\,\cos\vartheta\,\mathrm{d}\vartheta. \tag{51}$$

Substitution $v = \cos\vartheta$ and evaluation of the arising integral [19]

$$\int_0^1 x\,\sin\left(b\sqrt{1-x^2}\right)\,J_0(cx)\,\mathrm{d}x = \frac{b\sin\left(\sqrt{b^2+c^2}\right)}{(b^2+c^2)^{3/2}} - \frac{b\cos\left(\sqrt{b^2+c^2}\right)}{b^2+c^2}$$

allows us to obtain

$$T = \frac{aU_0 t^{\alpha-1} z}{\pi^2 (r^2+z^2)^{3/2}} \int_0^\infty \varrho\,E_{\alpha,\alpha}(-a\varrho^2 t^\alpha) \times \left[\sin\left(\varrho\sqrt{r^2+z^2}\right) - \varrho\sqrt{r^2+z^2}\cos\left(\varrho\sqrt{r^2+z^2}\right)\right]\mathrm{d}\varrho. \tag{52}$$

Similarly,

$$\Phi = -\frac{maU_0 t^{\alpha-1} z}{\pi^2 (r^2+z^2)^{3/2}} \int_0^\infty \frac{1}{\varrho}\,E_{\alpha,\alpha}(-a\varrho^2 t^\alpha) \times \left[\sin\left(\varrho\sqrt{r^2+z^2}\right) - \varrho\sqrt{r^2+z^2}\cos\left(\varrho\sqrt{r^2+z^2}\right)\right]\mathrm{d}\varrho. \tag{53}$$

Now we investigate several particular cases of the obtained solution. For classical thermoelasticity $\alpha = 1$ and $E_{1,1}(-x) = \mathrm{e}^{-x}$. Taking into account that

$$\int_0^\infty x^2\,\mathrm{e}^{-a^2x^2}\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{4a^3}\left(1 - \frac{b^2}{2a^2}\right)\exp\left(-\frac{b^2}{4a^2}\right), \quad a > 0,$$

and [18]

$$\int_0^\infty x\,\mathrm{e}^{-a^2x^2}\,\sin(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}b}{4a^3}\,\exp\left(-\frac{b^2}{4a^2}\right), \quad a > 0,$$

$$\int_0^\infty \frac{1}{x}\,\mathrm{e}^{-a^2x^2}\,\sin(bx)\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{erf}\left(\frac{b}{2a}\right), \quad a > 0,$$

$$\int_0^\infty \mathrm{e}^{-a^2x^2}\,\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2a}\,\exp\left(-\frac{b^2}{4a^2}\right), \quad a > 0,$$

we obtain

$$T = \frac{U_0 z}{8\pi^{3/2} a^{3/2} t^{5/2}}\,\exp\left(-\frac{r^2+z^2}{4at}\right), \tag{54}$$

$$\Phi = -\frac{maU_0 z}{2\pi R^{3/2}}\left[\operatorname{erf}\left(\frac{R}{2\sqrt{at}}\right) - \frac{R}{\sqrt{\pi at}}\,\exp\left(-\frac{R^2}{4at}\right)\right], \quad R = \sqrt{r^2+z^2}, \tag{55}$$

$$B = \frac{maU_0}{2\pi\xi}\left[\frac{1}{\sqrt{\pi at}\xi}\,\mathrm{e}^{-at\xi^2} - \operatorname{erfc}\left(\sqrt{at}\xi\right)\right]. \tag{56}$$

In the case of heat conduction with $\alpha = 1/2$

$$E_{1/2,\,1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-u^2-2ux}u\,\mathrm{d}u \tag{57}$$

and

$$T = \frac{U_0 z}{16\sqrt{2}\pi^2 a^{3/2}t^{7/4}}\int_0^\infty \frac{1}{v^{3/2}}\,\exp\left(-v^2-\frac{r^2+z^2}{8va\sqrt{t}}\right)\mathrm{d}v, \tag{58}$$

$$\begin{aligned}\Phi = &-\frac{maU_0 z}{\pi^{3/2}R^{3/2}\sqrt{t}}\int_0^\infty \left[\operatorname{erf}\left(\frac{R}{2^{3/2}t^{1/4}\sqrt{av}}\right)\right.\\ &\left.-\frac{R}{\sqrt{2\pi av}t^{1/4}}\,\exp\left(-\frac{R^2}{8a\sqrt{t}v}\right)\right]\mathrm{d}v,\end{aligned} \tag{59}$$

$$B = \frac{maU_0}{\pi^{3/2}\sqrt{t}\,\xi}\int_0^\infty v\mathrm{e}^{-v^2}\left[\frac{\mathrm{e}^{-2a\sqrt{t}v\xi^2}}{\sqrt{2\pi av}t^{1/4}\xi} - \operatorname{erfc}\left(\sqrt{2av}t^{1/4}\xi\right)\right]\mathrm{d}v. \tag{60}$$

Using the integral transform technique, similar results can be obtained for other types of boundary conditions for the time-fractional heat conduction equation.

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APPLICATION OF MODULAR COMPUTING TECHNIQUE FOR HIGH SPEED IMPLEMENTATION OF CYCLIC CONVOLUTION

MIKHAIL SELYANINOV

Abstract

This article is a continuation of research on the modular computing structures defined on the set of polynomials over finite rings of integers. Advantages of minimal redundant polynomial-scalar modular number system are demonstrated on the example of computing cyclic convolution of discrete signals. Methods of execution of ring arithmetical operations as well as coding and decoding operations are considered.

1. Introduction

At the present time, quality of the execution of information processing procedures in particular data domain is largely determined by the selected mathematical model of the organization of information processing and the information technology implemented on this basis. During the last years specialists in the field of the analysis and processing of digital information have an increased interest in parallel forms of information transform on the basis of modular computing structures (MCS) [1–3].

Usage of arithmetic of modular number systems (MNS) is especially effective, first of all, in such areas as digital signal processing (DSP), synthesis of fast algorithms for discrete orthogonal conversions, cryptography, error-correcting codes, etc. Possessing a maximum level of internal parallelism, the MCS represent a unique means of decomposition of computing processes into independent from each other elementary sub-processes defined on mathematical models with elements of small digit capacity.

The main advantages of the MCS are the high efficiency, which is achieved through parallelism of modular arithmetic (MA) algorithms and their pipelining using tabular method of calculation, the possibility of mathematical calculations with high accuracy, effectiveness of control of failure situations

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by means of correcting modular codes. Due to regularity and uniformity the MCS give maximum effect with the use of special VLSI.

2. Formulation of the problem

The most optimal range of applications of the MCS constitutes DSP systems, which have higher requirements for such features as performance, computational accuracy and fault tolerance. The spectrum of the modern applications of the MCS includes numerous high-speed implementations of the laborious computing procedures belonging to algorithmic kernels of DSP systems for one or another purpose.

In theory and applications of DSP the discrete linear system with constant parameters, whose mechanism of action is based on the calculation of discrete convolution, are of fundamental importance. Calculation of the convolution of two periodic sequences is a widely used task of DSP. For example, digital filtering is based on calculating a convolution of the input signal and the impulse response of the filter. Also, in a number of cryptography tasks there appears a necessity of multiplication of two numbers whose magnitudes exceed limits in which the hardware representation of operations on the basis of existing computing techniques is possible. In particular, the Schönhage-Strassen method [4] reduces multiplication of large integers to evaluation of convolution of the sequences associated with arrays of digits of their representation in a positional number system.

It should be noted that the calculation of convolution by the "direct" method requires the excessive computational cost. Existing techniques based on the discrete Fourier transform (DFT) allow us to reduce computational complexity for certain values of the length of the convolution due to the existence of fast algorithms for computing the DFT. The traditional calculation of the convolution by means of DFT for large lengths of convoluted sequences may lead to computational errors, sometimes significant. This is due to the fact that the values of basis functions of the DFT are irrational numbers and the calculations can be presented only with limited accuracy, in connection with finite digit length of a computer.

In recent years, experts in the field of DSP exhibit heightened interest in polynomial MA. It is caused by increment in this area of the methods which are based on polynomial transformations and, as a consequence, by sharp increase of an amount of operations over polynomials in synthesized computing procedures.

Application of polynomial MA allows us to carry out calculations of the large convolutions by replacing them by a sequence of short convolutions on the basis of special methods of multiplication of polynomials. This also allows us to implement a parallelization of computations at a level of micro-operations. If two initial sequences of discrete samples are represented in the polynomial form, then the cyclic convolution procedure can be reduced to the multiplication of polynomials.

The advantages and application features for the implementation of MNS procedures of such a class defined on the set of polynomials are demonstrated below by the example of computation of cyclic convolution of sequences $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$, where the elements of the output sequence are defined by the equation:

$$c_\nu = \sum_{u=0}^{n-1} a_{|\nu-u|_n} b_u \quad (\nu = 0, 1, \ldots, n-1) \tag{1}$$

3. Polynomial modular number systems

There are many scientific and applied researches demanding of processing the information presented in the form of polynomials. Operations over polynomials are very important in modern computer algebra, DSP, coding theory, cryptography, etc. At the same time, modular technology of parallel computing structures defined on polynomial ranges is of great interest.

Let us consider the set $\mathbf{Z}_m[x]$ of all polynomials of finite degree with coefficients from the ring $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$ of absolutely least residues modulo m, where m is a natural number, and the real argument x. In this case the technique of constructing a MNS [5] first of all requires the creation of the complete set of residues (CSR) with respect to selected pairwise relatively prime polynomial modules. The following theorem is true.

Theorem 1. *In the set* $\mathbf{Z}_m[x]$ *for any polynomial* $f(x)$ *and arbitrary polynomial modules* $p(x)$ *with the degree* $\deg p(x) \geq 1$ *there are unique elements* $q(x)$ *and* $r(x)$ *such that*

$$f(x) = q(x)\, p(x) + r(x) \quad (\deg r(x) < \deg p(x)). \tag{2}$$

Let $p(x)$ be any element of s-th degree from $\mathbf{Z}_m[x]$. Then, according to Theorem 1, the set of all residual $r(x)$ of division of $f(x)$ by $p(x)$ (see (2)), where $f(x)$ represents every element from the set $\mathbf{Z}_m[x]$, coincides with the set

$$\mathbf{Z}_m^s[x] = \left\{ A(x) = \sum_{j=0}^{s-1} a_j x^j \;\middle|\; (a_0, a_1, \ldots, a_{s-1}) \in (\mathbf{Z}_m \times \ldots \times \mathbf{Z}_m) \right\}, \tag{3}$$

where m and s are the fixed positive integers; $m \geq 2$. The cardinality of the set (3) is equal to $N = |\mathbf{Z}_m^s[x]| = m^s$. Thus, the set $\mathbf{Z}_m^s[x]$ is a CSR modulo $p(x)$. For the CSR of this type a special notation $\langle\cdot\rangle_{p(x)}$ is used, while the residue $r(x)$ modulo $p(x)$ over the polynomial $f(x)$ is designated as $\langle f(x)\rangle_{p(x)}$.

In the general case, the polynomial modular number system (PMNS) with pairwise relatively prime polynomial modules $p_1(x), p_2(x), \ldots, p_n(x)$ is induced by the isomorphic mapping

$$\phi:\ \langle\cdot\rangle_{P(x)} \to \langle\cdot\rangle_{p_1(x)} \times \langle\cdot\rangle_{p_2(x)} \times \ldots \times \langle\cdot\rangle_{p_n(x)},$$

where $P(x) = \prod\limits_{l=1}^{n} p_l(x)$. The isomorphism ϕ associates each polynomial $A(x) \in P(x)$ with the polynomial modular code (MC)

$$(a_1(x); a_2(x); \ldots; a_n(x)),$$

whose components are the residues $a_l(x) = \langle A(x)\rangle_{p_l(x)}$ $(l = 1, 2, \ldots, n)$ [6]. The set $\langle\cdot\rangle_{P(x)}$ is called the range of the PMNS.

The ring operations on polynomial modules $p_1(x), p_2(x), \ldots, p_n(x)$ over any two polynomials $A(x) = (a_1(x);\ a_2(x);\ \ldots;\ a_n(x))$ and $B(x) = (b_1(x);\ b_2(x);\ \ldots;\ b_n(x))$ $(a_l(x) = \langle A(x)\rangle_{p_l(x)}, b_l(x) = \langle B(x)\rangle_{p_l(x)}, l = 1, 2, \ldots, n)$ are executed independently for each module, i.e. according to the rule

$$\langle A(x) \circ B(x)\rangle =$$

$$\left(\langle a_1(x) \circ b_1(x)\rangle_{p_1(x)};\ \langle a_2(x) \circ b_2(x)\rangle_{p_2(x)};\ \ldots;\ \langle a_n(x) \circ b_n(x)\rangle_{p_n(x)}\right), \quad (4)$$

where $\circ \in \{+,\ -,\ \times\}$

Thus, both addition and multiplication of any two polynomials modulo $P(x)$ for their realizations require, respectively, n real additions and multiplications which can be executed in parallel in one modular clock tick. In the PMNS, all the operations (both modular (4) and non-modular) are performed in the ring $\mathbf{Z}_m$. This ring is called the scalar range or the numeric range of the PMNS. Decoding mapping assigning a polynomial $A(x)$ from the range $\langle\cdot\rangle_{P(x)}$ to a polynomial MC $(a_1(x),\ a_2(x),\ \ldots,\ a_n(x))$ is implemented by means of the Chinese reminder theorem [1, 7] which for the PMNS with modules $p_l(x)$ $(l = 1, 2, \ldots, n)$ gives

$$A(x) = \left\langle \sum_{l=1}^{n} P_l(x)\langle P_l(x)^{-1}A(x)\rangle_{p_l(x)} \right\rangle_{P(x)} =$$

$$= \sum_{l=1}^{n} P_l(x)\big\langle P_l(x)^{-1}A_l\big\rangle_{p_l(x)}, \quad (5)$$

where $P_l(x) = P(x)/p_l(x)$, $\left\langle P_l(x)^{-1}\right\rangle_{p_l(x)}$ is the residue which satisfies the equality $\left\langle P_l(x)\left\langle P_l(x)^{-1}\right\rangle_{p_l(x)}\right\rangle_{p_l(x)} = 1$.

The PMNS being the most appropriate for practical applications in the field of DSP have the polynomial modules $p_1(x), p_2(x), \ldots, p_n(x)$ which are the normalized polynomials of the first degree and the polynomial

$$P(x) = x^n - 1$$

has a factorization of the form

$$P(x) = \prod_{l=1}^{n}(x - r_l) \quad (r_l \in \mathbf{Z}_m; l = 1, 2, \ldots, n).$$

4. Implementation of discrete cyclic convolution in the PMNS

Let us consider the calculation of cyclic convolution of the sequences a_0, a_1, …, a_{n-1} and b_0, b_1, …, b_{n-1} according to formula (1). All the elements a_i and b_i $(i = 0, 1, \ldots, n-1)$ of the input sequences without loss of generality can be treated as integer variables taking values from the $\mathbf{Z}_m$. If the considered initial discrete sequences are presented in the polynomial form:

$$A(x) = a_{\nu-1}x^{\nu-1} + a_{\nu-2}x^{\nu-2} + \ldots + a_2x^2 + a_1x^1 + a_0$$

and

$$B(x) = b_{\nu-1}x^{\nu-1} + b_{\nu-2}x^{\nu-2} + \ldots + b_2x^2 + b_1x^1 + b_0,$$

then cyclic convolution operation can be reduced to the procedure of polynomials multiplication [8]. Thus, the realization of (1) is equivalent to the computation of the coefficients of the polynomial

$$C(x) = \sum_{\nu=0}^{n-1} c_\nu x^\nu = \left\langle \sum_{\nu=0}^{n-1} a_\nu x^\nu \sum_{\nu=0}^{n-1} b_\nu x^\nu \right\rangle_{x^n-1} = \langle A(x)\, B(x)\rangle_{x^n-1}\,.$$

The polynomials $A(x)$ and $B(x)$ are uniquely encoded in the PMNS, respectively, by sets of residues

$$A(x) = (a_1(x); a_2(x); \ldots; a_n(x)) \tag{6}$$

and

$$B(x) = (b_1(x); b_2(x); \ldots; b_n(x)) \tag{7}$$

corresponding to the selected polynomial modules $p_1(x)$, $p_2(x)$, …, $p_n(x)$ ($p_l(x) = x - r_l$ is the normalized polynomial of the first degree,

$$a_l(x) = \langle A(x)\rangle_{p_l(x)}\,;$$

$$b_l(x) = \langle B(x)\rangle_{p_l(x)}\,; \quad (l = 1, 2, \ldots, n).$$

The resultant polynomial

$$C(x) = (c_1(x); c_2(x); \dots; c_n(x)) \tag{8}$$

is obtained by multiplying of polynomials (6) and (7) in the PMNS

$$C(x) = \langle A(x)\, B(x) \rangle_{P(x)}$$

where $P(x) = \prod_{l=1}^{n} p_l(x) = x^n - 1$.

In this case, the components of the set of residues $(c_1(x), c_2(x), \dots, c_n(x))$ are the least residues of the division of products $a_l(x)\, b_l(x)$ by the corresponding polynomial modules

$$c_l(x) = \langle a_l(x)\, b_l(x) \rangle_{p_l(x)} \quad (l = 1, 2, \dots, n)$$

Reconstruction of the positional code of the polynomial $C(x)$ by its polynomial MC (8) is carried out in accordance with formula (5).

5. Processing of polynomial residues in minimal redundant MC

It follows from formula (4) that the efficiency level of PMNS arithmetic depends not only on analytical form of the modules $p_l(x)$ $(l = 1, 2, \dots, n)$ but also on the number system used for implementation of the computation over polynomial residues in the ring $\mathbf{Z}_m$. Since these calculations have a modular structure, then for encoding and processing of elements from the scalar range $\mathbf{Z}_m$ it is quite natural to use the real MNS with the modules $m_1, m_2, \dots, m_k$ and the range $M_k = \prod_{i=1}^{k} m_i$ for number representation [1, 7]. In this approach, the parameter m is equal to M_k, i.e. the ring $\mathbf{Z}_{M_k} = \{0, 1, \dots, M_k - 1\}$ is used as a numerical range of the PMNS. Such a PMNS with modular coding of elements of scalar range is called the polynomial-scalar MNS (PSMNS) [6].

Efficiency of computer arithmetic of PSMSS increases significantly when the minimal redundant modular coding of scalar elements is used. Minimal redundant encoding at the lower level allows us to optimize the execution of non-modular procedures [1, 7]. Such a PSMNS is called minimal redundant PSMNS. It is known that the principle of minimal redundant modular coding assumes that the set $\mathbf{Z}_{2M}^{-} = \{-M, -M + 1, \dots, M - 1\}$ (where $M = \prod_{i=0}^{k-1} m_i$, $m_k \geq m_0 + k - 2, m_0 \geq k - 2$, m_0 is additional natural module) is used as a scalar range of the PSMNS instead of the range $\mathbf{Z}_{M_k}$ [1], [6] and [7].

Thus, in this case the minimal redundant PSMNS is defined by the set of pairwise relatively prime normalized polynomials of the first degree

$$p_l(x) = x - r_l,\ (r_l \in \mathbf{Z}_{2M}^{-}, l = 1, 2, \ldots, n)$$

and the set of pairwise relatively prime natural modules $m_1,\ m_2,\ \ldots,\ m_k$.

In accordance with the above, an arbitrary polynomial $A(x) \in \langle\cdot\rangle_{P(x)}$ in minimal redundant PSMNS is encoded by a set of residues

$$(\alpha_{1,1},\ \alpha_{1,2}, \ldots,\ \alpha_{1,k};\ \alpha_{2,1},\ \alpha_{2,2}, \ldots,\ \alpha_{2,k};\ \ldots;\ \alpha_{n,1},\ \alpha_{n,2}, \ldots,\ \alpha_{n,k}), \quad (9)$$

where $\alpha_{l,i} = |A_l|_{m_i}$; $A_l = \langle A(x) \rangle_{p_l(x)} = |A(r_l)|_{M_k}$; the value $|X|_m$ denotes the least non-negative residue of dividing X by natural modulo m,

$$l = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, k.$$

Minimal redundant PSMNS are characterized by parallel structure both on the lower and upper levels of modular operations. In accordance with (4), the operations over any two polynomials $A(x)$ and $B(x)$ from the range $\langle\cdot\rangle_{P(x)}$ are executed by the rule

$$\begin{aligned}
&(\alpha_{1,1},\ \alpha_{1,2}, \ldots,\ \alpha_{1,k};\ \alpha_{2,1},\ \alpha_{2,2}, \ldots,\ \alpha_{2,k};\ \ldots;\ \alpha_{n,1},\ \alpha_{n,2}, \ldots, \alpha_{n,k}) \circ \\
&\circ(\beta_{1,1},\ \beta_{1,2}, \ldots, \beta_{1,k};\ \beta_{2,1},\ \beta_{2,2}, \ldots, \beta_{2,k};\ \ldots;\ \beta_{n,1},\ \beta_{n,2}, \ldots, \beta_{n,k}) = \\
&\qquad = (\ |\alpha_{1,1} \circ \beta_{1,1}|_{m_1},\ |\alpha_{1,2} \circ \beta_{1,2}|_{m_2}, \ldots,\ |\alpha_{1,k} \circ \beta_{1,k}|_{m_k}; \\
&\qquad |\alpha_{2,1} \circ \beta_{2,1}|_{m_1},\ |\alpha_{2,2} \circ \beta_{2,2}|_{m_2}, \ldots,\ |\alpha_{2,k} \circ \beta_{2,k}|_{m_k};\ \ldots \\
&\qquad |\alpha_{n,1} \circ \beta_{n,1}|_{m_1},\ |\alpha_{n,2} \circ \beta_{n,2}|_{m_2}, \ldots,\ |\alpha_{n,k} \circ \beta_{n,k}|_{m_k}\ ), \qquad (10)
\end{aligned}$$

where $A_{l,i} = |A(r_l)|_{m_i}$ and $B_{l,i} = |B(r_l)|_{m_i}$ are the digits of polynomial-scalar modular codes of the operands $A(x)$ and $B(x)$, respectively (see (9)), $\circ \in \{+,\ -,\ \times\}$.

One of the main advantages of the PSMNS is the unique possibility to calculate the sum, difference and especially the product of two polynomials in accordance with (10) in one modular clock tick. Thus, in this system the multiplication of any two polynomials modulo $P(x) = x^n - 1$ for its implementation requires only n multiplications executed in parallel. In contrast, in the case of traditional arithmetic in positional number system the polynomials multiplication in the ring $\langle\cdot\rangle_{P(x)}$ requires the executions of $n(n-1)$ real additions and n^2 real multiplications.

It is quite clear that the efficiency of applied methods for conversion of polynomials from positional number system to PSMNS and vice versa, as well as execution of other non-modular operations, can have significant influence on real effect of introducing the polynomial MA in practice. This problem is successfully solved using a minimal redundant modular coding of scalars from the range $\mathbf{Z}_{2M}^{-}$ [1, 7].

At first, let us consider forming of digits of polynomial-scalar MC. In particular, for calculating the digits $\alpha_{l,i}$ (see (9)) of arbitrary polynomial $A(x) = \sum_{\nu=0}^{n-1} a_\nu x^\nu$ from the range $\langle\cdot\rangle_{P(x)}$ ($a_\nu \in \mathbf{Z}^-_{2M}$) the following formula is used

$$\alpha_{l,i} = \left| \sum_{\nu=0}^{n-1} R_{\nu,\,l,\,i} \sum_{s=0}^{\mu-1} \left| F_s({a_\nu}^{(s)}) \right|_{m_i} \right|_{m_i},$$

where $R_{\nu,\,l,\,i} = |r_l^\nu|_{m_i}$; $F_s({a_\nu}^{(s)})$ are the additive components of λ-bit positional forms of coefficients a_ν:

$$a_\nu = \sum_{t=0}^{\lambda-1} a_{\nu,t} 2^t - a_{\nu,\,\lambda-1} 2^\lambda = \sum_{t=0}^{\lambda-2} a_{\nu,t} 2^t - a_{\nu,\,\lambda-1} 2^{\lambda-1} = \sum_{s=0}^{\mu-1} F_s(a_\nu^{(s)})$$

defined according to formulas

$$a_\nu^{(s)} = \sum_{t=0}^{\lambda_s-1} a_{\nu,\,q_s+t}\, 2^t \quad (s = 0, 1, \cdots, \mu-1)$$

$$F_s(a_\nu^{(s)}) = \begin{cases} a_{\nu,s}\, 2^{\,q_s} \text{ if } s = 0, 1, \cdots, \mu-2, \\ a_{\nu,\,\mu-1}\, 2^{\,q_{\mu-1}} - \lfloor a_{\nu,\,\mu-1} \,/\, 2^{\,q_{\mu-1}-1} \rfloor\, 2^\lambda \text{ if } s = \mu - 1; \end{cases}$$

$q_0 = 0, q_1, \cdots, q_{\mu-1}$ is the increasing sequence of integer values that specifies the partition of the binary additional code $(a_{\nu,\,\lambda-1}, a_{\nu,\,\lambda-2}, \cdots, a_{\nu,0})_2$ on $\mu \geq 1$ groups, sth of which contains $\lambda_s = q_{s+1} - q_s$ bits, $q_{\mu-1} \leq \lambda - 1$, $q_\mu = \lambda$ [1, 9]. Here $\lfloor y \rfloor$ denotes the integer part of a real number y.

In order to restore the positional representation of the polynomial $A(x)$ by its minimal redundant polynomial-scalar MC (9) at first it is necessary to compute the minimal redundant MC $(\alpha_1^{(\nu)},\ \alpha_2^{(\nu)}, \cdots\ \alpha_k^{(\nu)})$ of the coefficient a_ν for every $\nu = 0, 1, \cdots, n-1$:

$$\alpha_i^{(\nu)} = |a_\nu|_{m_i} = \left| \sum_{l=1}^{n} R_{l,\,i}^{(\nu)}\, \alpha_{l,i} \right|_{m_i} \qquad (i = 1, 2, \cdots, k),$$

where $R_{l,\,i}^{(\nu)} = \left| n^{-1}\, r_l^{-\nu} \right|_{m_i}$ [6]. After that, the positional code of the coefficient a_ν can be formed by its MC in accordance with the formula

$$a_\nu = \sum_{i=1}^{k-1} M_{i,k-1} \left| M_{i,k-1}^{-1} \alpha_i^{(\nu)} \right|_{m_i} + I(a_\nu) M_{k-1},$$

where $M_{i,\,k-1} = M_{k-1}/m_i$, $M_{k-1} = \prod_{j=1}^{k-1} m_j$, $I(a_\nu)$ is the interval index of integer a_ν defined by the following calculating expressions [1, 7]

$$I(a_\nu) = \begin{cases} \hat{I}_k(a_\nu) \text{ if } \hat{I}_k(a_\nu) < m_0, \\ \hat{I}_k(a_\nu) - m_k \text{ if } \hat{I}_k(a_\nu) > m_k - m_0 - k + 2; \end{cases}$$

$$\hat{I}_k(a_\nu) = \left| \sum_{i=1}^{k} R_{i,k}(\alpha_i^{(\nu)}) \right|_{m_k}$$

$$R_{i,k}(\alpha_i^{(\nu)}) = \left| \frac{\left| M_{i,k-1}^{-1}\, \alpha_i^{(\nu)} \right|_{m_i}}{M_{k-1}} \right|_{m_k} \quad (i \neq k); \quad R_{k,k}(\alpha_k^{(\nu)}) = \left| \frac{\alpha_k^{(\nu)}}{M_{k-1}} \right|_{m_k}.$$

Thus, the efficiency of the PSMNS computer arithmetic is significantly increased due to the optimization of the non-modular procedures when using the minimal redundant coding at the lower level [1, 7]. Therefore, the minimal redundant PSMNS potentially takes the priority position in the field of computer applications.

The proposed developments allow us to create effective DSP systems using the minimally redundant PSMNS with sufficiently simple implementation. In these systems at the upper level the normalized polynomials of the first degree is used as a bases, whereas at the lower level the elements of scalar range is represented in minimal redundant MC.

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K-CONTINUITY PROBLEM OF K-SUPERQUADRATIC SET-VALUED FUNCTIONS

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Abstract

In this paper we study K-superquadratic set-valued functions.We will present here some connections between K-boundedness of K-superquadratic set-valued functions and K-semicontinuity of multifunctions of this kind.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function F is called superquadratic, if it fulfils inequality

$$2F(x) + 2F(y) \leq F(x + y) + F(x - y), \qquad x, y \in X. \tag{1}$$

If the sign "$\leq$" in (1) is replaced by "$\geq$", then F is called subquadratic. The continuity problem of functions of this kind was considered in [2]. This problem was also considered in the class of set-valued functions. In this case F is called subquadratic set-valued function, if it satisfies inclusion

$$F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \qquad x, y \in X \tag{2}$$

and superquadratic set-valued function, if it satisfies inclusion defined in such a form:

$$2F(x) + 2F(y) \subset F(x + y) + F(x - y), \qquad x, y \in X. \tag{3}$$

For usual (i.e. single-valued) functions the properties of subquadratic and superquadratic functions are quite analogous and, in view of the fact that if a function F is subquadratic, then the function $-F$ is superquadratic and conversely, it is not necessary to investigate functions of these two kinds individually.

In the case of set-valued functions the situation is different. Even if properties of subquadratic and superquadratic set-valued functions are similar, we have to prove them separately.

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If the sign "$\subset$" in the inclusions above is replaced by "$=$", then F is called quadratic set-valued function. The class of quadratic set-valued functions is an important subclass of the class of subquadratic and superquadratic set-valued functions. Quadratic set-valued functions have already extensive bibliography (see D. Henney [1], K. Nikodem [4] and W. Smajdor [5]). The continuity problem of subquadratic and superquadratic set-valued functions was considered in [6] and [7].

If we enlarge the space of values of a set-valued function F by a cone K we can consider K-superquadratic set-valued functions, that is solutions of the inclusion

$$F(x+y)+F(x-y)\subset 2F(x)+2F(y)+K,\qquad x,y\in X. \tag{4}$$

The concept of K-superquadraticity is related to real-valued superquadratic functions. Note, in the case when F is a single-valued real function and $K=[0,\infty)$, we obtain the standard definition of superquadratic functions (1).

Similarly, if a set-valued function F satisfies the following inclusion

$$2F(x)+2F(y)\subset F(x+y)+F(x-y)+K,\qquad x,y\in X \tag{5}$$

then it is called K-subquadratic. The K-continuity problem of multifunction of this kind was considered in [8]. It has been proved there that a K-subquadratic set-valued function F defined on 2-divisible topological group X with non-empty, compact and convex values in a locally convex topological vector space Y, which is K-continuous at zero and locally K-bounded in X, is K-continuous everywhere in X.

In this paper we shall consider similar problem for K-superquadratic set-valued functions. Likewise as in functional analysis we can look for connections between K-boundedness and K-semi-continuity of set-valued functions of this kind.

Assuming $K=\{0\}$ in (4) and (5), we obtain the inclusions (2) and (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y and $cc(Y)-$the family of all compact and convex members of $n(Y)$. The term *set-valued function* will be abbreviated to the form s.v.f.

Recall that a set $K\subset Y$ is called a cone iff $K+K\subset K$ and $sK\subset K$ for all $s\in(0,\infty)$.

Definition 1. (cf. [3]) *A cone K in a topological vector space Y is said to be a normal cone iff there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W=(W+K)\cap(W-K)$$

for all $W\in\mathfrak{W}$.

Definition 2. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper semicontinuous (abbreviated K-u.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower semicontinuous (abbreviated K-l.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-continuous at $x_0 \in X$ iff it is both K-u.s.c. and K-l.s.c. at x_0. It is said to be K-continuous iff it is K-continuous at each point of X.*

Note that in the case where $K = \{0\}$ the K-continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In this paper we will use the following lemma.

Lemma 1. (cf. [8]) *Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A+C \subset B+C+K$. If B is convex and C is bounded, then $A \subset \overline{B+K}$.*

2. The main result

In the proof of the main theorem we will often use four known lemmas (see Lemma 1.1, Lemma 1.3, Lemma 1.6 and Lemma 1.9 in [9]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s+t)A = sA + tA$ holds for every $s, t \geq 0$ or (s,t<0). The second lemma says that in a real vector space Y for two convex subsets A, B the set $A+B$ is also convex. The next lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A+B$ is closed. For any sets $A, B \subset Y$, where Y denotes the same space as above, the inclusion $\overline{A} + \overline{B} \subset \overline{A+B}$ holds and the equality holds if and only if the set $\overline{A} + \overline{B}$ is closed.

Notice that for the cone K the following remark holds.

Remark 1. *Let Y be a real topological vector space. If K is a closed cone, then it is a cone with zero.*

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$. An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower bounded at a point $x \in X$ iff there exists a neighbourhood U_x of zero in X such that F is K-lower bounded on a set $x + U_x$*

Definition 6. *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$. An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper bounded at a point $x \in X$ iff there exists a neighbourhood U_x of zero in X such that F is K-upper bounded on a set $x + U_x$*

Definition 7. *An s.v.f. $F\colon X \to n(Y)$ is said to be locally K-bounded in X iff it is both K-upper and K-lower bounded at every point $x \in X$.*

Definition 8. *We say that 2-divisible topological group X has the property $\left(\frac{1}{2}\right)$ iff for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.*

For the K-superquadratic set-valued functions the following theorem holds.

Theorem 1. *Let X be a 2-divisible topological group with property $\left(\frac{1}{2}\right)$, Y – locally convex topological real vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K-bounded in X, then it is K-u.s.c. in X.*

Proof. Suppose that F is not K-u.s.c. at a point $z \in X$, i.e. there exists a neighbourhood V of zero in Y such that for every neighbourhood U of zero in X we can find $x_u \in U$ for which

$$F(z + x_u) \not\subset F(z) + V + K.$$

Take a balanced convex neighbourhood W of zero in Y such that

$$W \subset V$$

and

$$\overline{F(z) + W + K} \subset F(z) + V + K.$$

Then also

$$F(z + x_u) \not\subset \overline{F(z) + W + K}. \tag{6}$$

Let a neighbourhood U of zero in X be arbitrarily fixed. Suppose that

$$F(z + x_u) + 2^k\left(2^k - 1\right)F(x_u) \not\subset \overline{F\left(z + \left(1 - 2^k\right)x_u\right) + 2^kW + K} \tag{7}$$

for some $k \in \mathbb{N} \cup \{0\}$. The proof of (7) runs by induction. For $k = 0$ condition (7) holds with respect to (6). Putting $y = x$ in (4) and using condition $F(0) = \{0\}$, we have

$$F(2x) \subset 4F(x) + K.$$

An easy induction shows

$$F(2^n x) \subset 4^n F(x) + K \tag{8}$$

for $x \in X$ and for all positive integers $n \in \mathbb{N}$. By K-superquadraticity of F and (8), we have

$$\begin{aligned} &F\left(z + (1 - 2^{k+1})x_u\right) + F(z + x_u) = \\ &= F\left(z + x_u - 2^k x_u - 2^k x_u\right) + F\left(z + x_u - 2^k x_u + 2^k x_u\right) \subset \\ &\subset 2F\left(z + x_u - 2^k x_u\right) + 2F\left(2^k x_u\right) + K \subset \\ &\subset 2F\left(z + (1 - 2^k)x_u\right) + 2^{2k+1}F(x_u) + K. \end{aligned} \tag{9}$$

In view of the fact that for any sets $A, B \subset Y$, $\overline{A} + \overline{B} \subset \overline{A + B}$ we get

$$\begin{aligned} &\overline{F\left(z + (1 - 2^k)x_u\right) + 2^k W + K} + K \subset \\ &\subset \overline{F\left(z + (1 - 2^k)x_u\right) + 2^k W + K} \end{aligned}$$

and, consequently,

$$\begin{aligned} &\overline{\overline{F(z + (1 - 2^k)x_u) + 2^k W + K} + K} \subset \\ &\subset \overline{F(z + (1 - 2^k)x_u) + 2^k W + K}. \end{aligned} \tag{10}$$

By (7) and (10), we obtain

$$F(z + x_u) + 2^k \left(2^k - 1\right) F(x_u) \not\subseteq \overline{\overline{F(z + (1 - 2^k)x_u) + 2^k W + K} + K}.$$

Notice that for a cone K the equality $aK = K$ holds for every $a \in (0, \infty)$. Hence,

$$\begin{aligned} &2F(z + x_u) + 2^{k+1} \left(2^k - 1\right) F(x_u) \not\subseteq \\ &\not\subseteq \overline{\overline{2F(z + (1 - 2^k)x_u) + 2^{k+1} W + K} + K}. \end{aligned} \tag{11}$$

By (11) and Lemma 1,

$$\begin{aligned} &2F(z + x_u) + 2^{k+1} \left(2^k - 1\right) F(x_u) + 2^{2k+1} F(x_u) \not\subseteq \\ &\not\subseteq \overline{2F(z + (1 - 2^k)x_u) + 2^{k+1} W + K} + 2^{2k+1} F(x_u) + K. \end{aligned}$$

In view of Remark 1, K is a cone with zero. Therefore by above,

$$2F(z+x_u)+2^{k+1}\left(2^k-1\right)F(x_u)+2^{2k+1}F(x_u)+K\nsubseteq \tag{12}$$

$$\nsubseteq \overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)+K.$$

In view of the fact that the sum of closed and compact sets is closed and for any sets $A,B\subset Y$, $\overline{A}+\overline{B}=\overline{A+B}$, in the case where $\bar{A}+\bar{B}$ is a closed set, we get

$$\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)= \tag{13}$$

$$=\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)}.$$

Since K is a cone, by (9), we obtain

$$\overline{F(z+(1-2^{k+1})x_u)+F(z+x_u)+2^{k+1}W+K}\subset \tag{14}$$

$$\subset\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)}.$$

Since F has closed values, we get

$$F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K\subset \tag{15}$$

$$\subset\overline{F(z+(1-2^{k+1})x_u)+F(z+x_u)+2^{k+1}W+K}+K.$$

Consequently, by (12–15) we conclude

$$2F(z+x_u)+2^{k+1}\left(2^k-1\right)F(x_u)+2^{2k+1}F(x_u)+K\nsubseteq$$

$$\nsubseteq F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K.$$

By convexity of the sets $F(x_u)$ i $F(z+x_u)$, we obtain

$$F(z+x_u)+F(z+x_u)+2^{k+1}\left(2^{k+1}-1\right)F(x_u)+K\nsubseteq$$

$$\nsubseteq F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K.$$

Therefore,

$$F(z+x_u)+2^{k+1}\left(2^{k+1}-1\right)F(x_u)\nsubseteq$$

$$\nsubseteq\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}.$$

We have proved that (7) holds for every neighbourhood U of zero in X and $k=0,1,2...$

Since K is a normal cone, there exists a base $\mathfrak{W}$ of neighbourhoods of zero in Y such that $M=(M+K)\cap(M-K)$ for all $M\in\mathfrak{W}$. We can choose $W_1\in\mathfrak{W}$ and balanced neighbourhood W_2 of zero in Y such that

$$W_2\subset W_1\subset W.$$

Because F is K-lower bounded on a neighbourhood of z, there exists a neighbourhood U_0 of zero in X and a bounded set $B_1 \subset Y$ such that

$$F(z+t) \subset B_1 + K, \qquad t \in U_0.$$

Since the set B_1 is bounded, there exists $\lambda_1 > 0$ such that

$$B_1 \subset \frac{1}{\lambda_1} W_2.$$

Therefore, by above,

$$F(z+t) \subset \frac{1}{\lambda_1} W_2 + K, \qquad t \in U_0.$$

Similarly, since F is K-upper bounded on a neighbourhood of z, there exists a neighbourhood U_1 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(z+t) \subset B_2 - K, \qquad t \in U_1.$$

Since the set B_2 is bounded, there exists $\lambda_2 > 0$ such that

$$B_2 \subset \frac{1}{\lambda_2} W_2.$$

Therefore, by above,

$$F(z+t) \subset \frac{1}{\lambda_2} W_2 - K, \qquad t \in U_1.$$

Let $\lambda := \min\{\lambda_1, \lambda_2\}$. Since W_2 is a balanced set, we get

$$F(z+t) \subset \frac{1}{\lambda} W_2 + K \subset \frac{1}{\lambda} W_1 + K, \qquad t \in U_0 \tag{16}$$

and

$$F(z+t) \subset \frac{1}{\lambda} W_2 - K \subset \frac{1}{\lambda} W_1 - K, \qquad t \in U_1. \tag{17}$$

By (16) and (17), we obtain

$$F(z+t) \subset \left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right), \qquad t \in U_0 \cap U_1. \tag{18}$$

Because of $W_1 \in \mathfrak{W}$, we have

$$\left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right) = \frac{1}{\lambda} W_1$$

and, consequently, the following inclusion holds

$$F(z+t) \subset \frac{1}{\lambda} W \tag{19}$$

for every $t \in U_0 \cap U_1$.

Let $k \in \mathbb{N}$ be so large that

$$2^k > \frac{3}{\lambda}. \tag{20}$$

Let U be a symmetric neighbourhood of zero in X such that $U+U \subset U_0 \cap U_1$ and $\frac{1}{2}U \subset U$. Consider two sets $\frac{1}{2^k}U$ i $\frac{1}{\lambda 2^k(2^k-1)}W$. Since F is K-u.s.c. at zero and $F(0)=\{0\}$, there exists a symmetric neighbourhood U_2 of zero in X such that

$$U_2 \subset \frac{1}{2^k}U \subset U \tag{21}$$

and

$$F(t) \subset \frac{1}{\lambda 2^k(2^k-1)}W + K, \qquad t \in U_2. \tag{22}$$

There exists $x_u \in U_2$ such that (7) holds. By (21),

$$\left(1-2^k\right)x_u = x_u - 2^k x_u \in U_2 - U \subset U + U \subset U_0 \cap U_1 \tag{23}$$

and by (22),

$$F(x_u) \subset \frac{1}{\lambda 2^k(2^k-1)}W + K. \tag{24}$$

Let $a \in F\left(z+(1-2^k)x_u\right)$, $b \in F(z+x_u)$ i $c \in F(x_u)$. By (19), (20), (23) and (24), we obtain

$$b + 2^k\left(2^k-1\right)c - a \in \frac{1}{\lambda}W + \frac{1}{\lambda}W + K + \frac{1}{\lambda}W \subset 2^k W + K.$$

Therefore,

$$b + 2^k\left(2^k-1\right)c \in F\left(z+(1-2^k)x_u\right) + 2^k W + K.$$

We have proved that

$$F(z+x_u) + 2^k\left(2^k-1\right)F(x_u) \subset F\left(z+(1-2^k)x_u\right) + 2^k W + K,$$

which contradicts (7). □

This article is an introduction to the discussion on the K-continuity problem for K-superquadratic set-valued functions. In the theory of K-subquadratic and K-superquadratic set-valued functions an important role is played by theorems giving possibly weak conditions under which such multi-functions are K-continuous.

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K-CONTINUITY OF K-SUBQUADRATIC SET-VALUED FUNCTIONS

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Abstract

Let $X = (X, +)$ be an arbitrary topological group. A set-valued function $F\colon X \to n(Y)$ is called K-subquadratic if

$$2F(s) + 2F(t) \subset F(s+t) + F(s-t) + K,$$

for all $s,\, t \in X$, where Y denotes a topological vector space and where K is a cone in this space.

In this paper the K-continuity problem of multifunctions of this kind will be considered with respect to weakly K-boundedness. The case where $Y = \mathbb{R}^N$ will be considered separately.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function F, is called subquadratic, if it fulfils inequality

$$F(x+y) + F(x-y) \le 2F(x) + 2F(y), \qquad x, y \in X. \tag{1}$$

If the sign "$\le$" in (1) is replaced by "$\ge$" then F is called superquadratic. The continuity problem of functions of this kind was considered in [1]. This problem can be also considered in the class of set-valued functions. Then we have two inclusions

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y), \qquad x,\; y \in X \tag{2}$$

and

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y), \qquad x,\; y \in X. \tag{3}$$

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where $F\colon X \to n(Y)$ and where Y denotes a topological vector space. The continuity problem of set-valued functions defined by inclusions (2) and (3) was considered in [4] and [5].

Adding a cone K in the space of values let us consider a K-subquadratic set-valued function F, that is solution of the inclusion

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y) + K, \qquad x, y \in X \tag{4}$$

which is defined on 2-divisible topological group X with non-empty, compact and convex values in a locally convex topological vector space Y. The K-continuity problem of multifunctions of this kind was considered in [6]. Here the K-continuity problem of K-subquadratic set-valued functions will be considered with respect to the weakly K-boundedness. In the last part of this paper we will present some conditions which imply K-continuity of K-subquadratic multifunctions which values are in $n(\mathbb{R}^N)$.

The concept of K-subquadraticity is related to real-valued subquadratic functions. In case when F is a real single-valued function and $K = [0, \infty)$, we obtain the standard definition of subquadratic functionals (1). Assuming $K = \{0\}$ in (4) we obtain the inclusion (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y, $cc(Y)$ – the family of all compact and convex members of $n(Y)$, $B(Y)$ – the family of all bounded members of $n(Y)$ and $Bcc(Y)$ – the family of all bounded, compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

First of all we shall present some definitions for the sake of completeness. Recall that a set $K \subset Y$ is called a cone if $K + K \subset K$ and $sK \subset K$ for all $s \in (0, \infty)$.

Definition 1. (cf. [2]) *A cone K in a topological vector space Y is said to be a normal cone if there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W = (W + K) \cap (W - K)$$

for all $W \in \mathfrak{W}$.

Definition 2. (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper semicontinuous (abbreviated K-u.s.c.) at $x_0 \in X$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower semicontinuous (abbreviated K-l.s.c.) at $x_0 \in X$ if for every neighbourhood V*

of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-continuous at $x_0 \in X$ if it is both K-u.s.c. and K-l.s.c. at x_0. It is said to be K-continuous if it is K-continuous at each point of X.*

Note that in the case where $K = \{0\}$ the K-continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In our proofs we use known following lemma.

Lemma 1. (cf. [6]) *Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A + C \subset B + C + K$. If B is convex and C is bounded then $A \subset \overline{B + K}$.*

In our proofs we will also use two known lemmas (see Lemma 1.6 and Lemma 1.5 in [2]). The first lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A + B$ is closed. The second lemma says that for any bounded sets $A, B \subset Y$, where Y denotes the same space as above, the set $A + B$ is bounded.

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$.*

Definition 6. *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$.*

Definition 7. *An s.v.f. $F\colon X \to n(Y)$ is said to be locally K-bounded in X if for every $x \in X$ there exists a neighbourhood U_x of zero in X such that F is K-lower and K-upper bounded on a set $x + U_x$.*

2. The main result connected with weakly K-boundedness

Let us introduce the following definitions:

Definition 8. *An s.v.f. $F\colon X \to n(Y)$ is said to be weakly K-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that*

$$F(x) \cap (B + K) \neq \emptyset$$

for all $x \in A$.

Definition 9. *An s.v.f. $F\colon X \to n(Y)$ is said to be weakly K-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that*

$$F(x) \cap (B - K) \neq \emptyset$$

for all $x \in A$.

Definition 10. *An s.v.f. $F\colon X \to n(Y)$ is said to be locally weakly K-bounded in X if for every $x \in X$ there exists a neighbourhood U_x of zero in X such that F is weakly K-lower and weakly K-upper bounded on a set $x + U_x$.*

Clearly, if F is K-upper (K-lower) bounded on a set A, then it is weakly K-upper (K-lower) bounded on a set A. In the case of single-valued functions these definitions coincide.

Definition 11. *We say that 2-divisible topological group X has the property $\left(\frac{1}{2}\right)$ if for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.*

For the K-subquadratic set-valued functions the following theorem holds.

Theorem 1. (cf. [6]) *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y – locally convex topological vector space and a subset K of Y – a closed normal cone. If a K-subquadratic s.v.f. $F\colon X \to cc(Y)$ is K-continuous at zero, locally K-bounded in X and $F(0) = \{0\}$, then it is K-continuous in X.*

Lemma 2. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y – topological vector space and $K \subset Y$ a cone. Let $F\colon X \to B(Y)$ be a K-subquadratic s.v.f. such that $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset F(x) + K \tag{5}$$

for all $x \in X$.

If F is K-lower bounded at zero and G is locally weakly K-upper bounded in X, then F is locally K-lower bounded in X.

Proof. Let $x \in X$. There exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x - t) \cap (B_1 - K) \neq \emptyset, \quad t \in U_1,$$

which implies that

$$0 \in G(x - t) - B_1 + K \tag{6}$$

for all $t \in U_1$. Since F is K-lower bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 + K, \quad t \in U_2. \tag{7}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (5), (6) i (7), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1+K \subset F(x+t)+F(x-t)-B_1+K \subset$$
$$\subset \frac{1}{2}F(2x)+\frac{1}{2}F(2t)-B_1+K \subset \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1+K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B}+K, \quad t \in \widetilde{U},$$

which means that F is locally K-lower bounded in X. □

Lemma 3. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y topological vector space and $K \subset Y$ a cone. Let $F\colon X \to B(Y)$ be a K-subquadratic s.v.f. such that $F(0)=\{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset F(x)-K \tag{8}$$

for all $x \in X$.

If F is K-upper bounded at zero and G is locally weakly K-lower bounded in X, then F is locally K-upper bounded in X.

Proof. Let $x \in X$. Since G is weakly K-lower bounded in x, then there exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x-t) \cap (B_1+K) \neq \emptyset, \quad t \in U_1,$$

which implies that

$$0 \in G(x-t)-B_1-K \tag{9}$$

for all $t \in U_1$. Since F is K-upper bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2-K, \quad t \in U_2. \tag{10}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (8), (9) i (10), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1-K \subset F(x+t)+F(x-t)-B_1-K \subset$$
$$\subset \frac{1}{2}F(2x)+\frac{1}{2}F(2t)-B_1-K \subset \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1-K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B}-K, \quad t \in \widetilde{U},$$

which means that F is locally K-upper bounded in X. □

As an immediate consequence of Lemma 2 and Lemma 3 we obtain the following lemma.

Lemma 4. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y topological vector space and $K \subset Y$ a cone with zero. Let $F\colon X \to B(Y)$ be a K-subquadratic s.v.f. such that $F(0) = \{0\}$. If F is K- bounded at zero and locally weakly K-bounded in X, then it is locally K-bounded in X.*

Let us note, that Theorem 1, Lemma 2 and Lemma 3 yield directly the following result.

Theorem 2. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset (F(x) - K) \cap (F(x) + K)$$

for all $x \in X$.

If F is K-bounded at zero and K-continuous at zero, G is locally weakly K-bounded in X, then F is K-continuous everywhere in X.

Remark 1. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$.*

If F is K-continuous at zero, K- bounded at zero and locally weakly K-bounded in X, then it is K-continuous in X.

Proof. Note that a closed cone is a cone with zero. Then the following inclusion

$$F(x) \subset (F(x) - K) \cap (F(x) + K)$$

holds for all $x \in X$. Using Theorem 2 for $G = F$ we end the proof. $\square$

3. The case $Y = \mathbb{R}^N$

Now we consider the case where the space of values is $n(\mathbb{R}^N)$. It is known that for K-subquadratic set-valued functions the following lemma holds.

Lemma 5. (cf. [6]) *Let X be a 2-divisible topological group, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. If a K-subquadratic s.v.f. $F\colon X \to cc(Y)$ is K-continuous at zero, $F(0) = \{0\}$ and locally K-lower bounded in X, then it is K-u.s.c. in X.*

In this part of the paper Y will be denote $\mathbb{R}^N$.

Theorem 3. *Let X be a 2-divisible topological group and K be a closed normal cone in Y. Let $F\colon X \to cc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$. If F is K-continuous at zero and locally K-lower bounded in X, then it is K-continuous in X.*

Proof. By Lemma 5 F is K-u.s.c. in X. Now we will show that F is K-l.s.c. in X. Let $x_0 \in X$ and let V be a neighbourhood of zero in Y. There exists a convex neighbourhood W of zero in Y such that the set $\overline{W}$ is compact with $3\overline{W} \subset V$. Since F is K-u.s.c. at x_0 then there exists a symmetric neighbourhood U of zero in X such that

$$F(x_0 + t) \subset F(x_0) + W + K, \tag{11}$$

$$F(x_0 - t) \subset F(x_0) + W + K, \tag{12}$$

for all $t \in U$.
Since F is K-l.s.c. at zero and $F(0) = \{0\}$, there exists a neighbourhood U_0 of zero in X such that

$$\{0\} \subset F(t) + W + K \qquad t \in U_0. \tag{13}$$

Consider a symmetric neighbourhood $\widetilde{U}$ of zero in X with $\widetilde{U} \subset U \cap U_0$. Let $t \in \widetilde{U}$. Using (12) i (13), we obtain

$$F(x_0)+\{0\} \subset F(x_0)+F(t)+W+K \subset \frac{1}{2}F(x_0+t)+\frac{1}{2}F(x_0-t)+W+K \subset$$

$$\subset \frac{1}{2}F(x_0 + t) + \frac{1}{2}F(x_0) + \frac{3}{2}\overline{W} + K$$

By convexity of the set $F(x_0)$, we have

$$\frac{1}{2}F(x_0) + \frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0) + \frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{W} + K.$$

Note that the set $\frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{W}$ is convex and compact. Therefore, the set $\frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{W} + K$ is closed. Using Lemma 1

$$\frac{1}{2}F(x_0) \subset \overline{\frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{W} + K} = \frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{W} + K,$$

and consequently

$$F(x_0) \subset F(x_0 + t) + 3\overline{W} + K \subset F(x_0 + t) + V + K,$$

for all $t \in \widetilde{U}$. It means that F is K-l.s.c. in X. □

Theorem 4. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$ and K be a closed normal cone in $\mathbb{R}^N$. Let $F\colon X \to cc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$. If F is K-continuous at zero and locally weakly K-upper bounded in X, then it is K-continuous in X.*

Proof. Let V be a bounded neighbourhood of zero in Y. Since F is K-u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood U of zero in X such that

$$F(t) \subset V + K, \qquad t \in U.$$

It means that F is K-lower bounded at zero. By Lemma 2 (with $G = F$), F is locally K-lower bounded in X. Applying Theorem 3, F is K-continuous in X. $\square$

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ABOUT DIFFERENTIABILITY AND VBG_* CLASS

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Abstract

Let X be a finite dimensional real Banach space. We show that if the contingent of the curve $\Gamma\colon (a,b) \to X$ fulfils some conditions then each parametrization of that curve is VBG_*. Stanisław Saks proved that each VBG_* function is differentiable at a set of full Lebesgue measure. The result of this paper is a partial converse of that theorem.

1. Introduction

We will present a generalization of the concepts of functions of bounded variation in the restricted sense (VB_*) and of generalized bounded variation in the restricted sense (VBG_*) in the case of functions of a real variable that takes values in a real normed space. Let us recall first these definitions in the case of real-valued functions.

Definition 1. [2], [5] *If $F\colon [a,b] \to \mathbb{R}$ and $[\alpha,\beta] \subset [a,b]$, then the value*

$$\sup\big\{|F(x) - F(y)|\colon x \in [\alpha,\beta], y \in [\alpha,\beta]\big\}$$

is called an oscillation of the mapping F on the interval $[\alpha,\beta]$ and is denoted by $\omega\big(F,[\alpha,\beta]\big)$.

Definition 2. [2], [5] *If $F\colon [a,b] \to \mathbb{R}$ and $E \subset [a,b]$ then a mapping F is called of bounded variation in the restricted sense on the set E, or simply, is of VB_* on E, if*

$$\sup \sum_k \omega\big(F,[a_k,b_k]\big) < \infty,$$

where $([a_k,b_k])_{k\in\mathbb{N}}$ is any sequence of non-overlapping intervals such that $a_k \in E$, $b_k \in E$. The number $\sup\sum_k \omega\big(F,[a_k,b_k]\big)$ is denoted by $V_E F$.

Definition 2 can be generalized on the case of the mapping F with value in a real normed space X.

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Definition 3. *Let X be a real normed space and $\|\cdot\|$ be the norm in X. By the oscillation of a mapping $F\colon [a,b]\to X$ on $[\alpha,\beta]\subset[a,b]$ we call the value*

$$\sup\bigl\{\|F(x)-F(y)\|\colon x\in[\alpha,\beta], y\in[\alpha,\beta]\bigr\}.$$

This oscillation will be denoted by the symbol $\omega\bigl(F,[\alpha,\beta]\bigr)$.

Definition 4. *Let X be a real normed space and $E\subset[a,b]$. We say that a mapping $F\colon [a,b]\to X$ is VB_* on the set E, and denote $F\in VB_*(E)$, if*

$$\sup\sum_k \omega\bigl(F,[a_k,b_k]\bigr)<\infty,$$

where $([a_k,b_k])_{k\in\mathbb{N}}$ is any sequence of non-overlapping intervals such that $a_k\in E$, $b_k\in E$. The value $\sup\sum_k\omega\bigl(F,[a_k,b_k]\bigr)$ is denoted by $V_E F$.

Now we assume that dimension of X is finite. Observe that the fact that F is VB_* on some set is independent of the choice of a norm in X. Let $F\colon [a,b]\to X$ and $e=(e_1,\dots,e_n)$ be a base of the space X. Then

$$F=\sum_{i=1}^{n} F_i e_i.$$

Mappings F_i are called coordinates of the mapping F with respect to the base e. We also shall use denotation $F=(F_1,\dots,F_n)$. Straightforward calculations prove the next lemma.

Lemma 1. *If X is a finite dimensional real normed space, $F\colon [a,b]\to X$ and $E\subset[a,b]$, then:*

1. *If F is VB_* on E then for each base $e=(e_1,\dots,e_n)$ of the space X mappings F_i are VB_* on E for each $i\in\{1,\dots,n\}$.*
2. *If there exists a base $e=(e_1,\dots,e_n)$ of the space X for which mappings F_i, $i\in\{1,\dots,n\}$, are VB_* on the set E then $F\in VB_*(E)$.*

Definition 5. [2], [5] *Let $E\subset[a,b]$. We say that a mapping $F\colon [a,b]\to\mathbb{R}$ is of generalized bounded variation in the restricted sense on E, or simply, is VBG_* on the set E, and denote $F\in VBG_*(E)$, if E is a countable union of sets on each of which the mapping F is VB_*.*

We can generalize this definition in the following way:

Definition 6. *Let X be a real normed space, $E\subset[a,b]$. We will say that a mapping $F\colon [a,b]\to X$ is VBG_* on E, and denote $F\in VBG_*(E)$, if E is a countable union of sets such that for each of them F is VB_*.*

The proof of the next lemma is technical, we shall omit it.

Lemma 2. *Let X be a real normed space, $\dim X=n$, $F\colon [a,b]\to X$ and $E\subset[a,b]$. Then*

(1) *If F is VBG_* on E then for each base $e = (e_1, \dots, e_n)$ of the space X mappings F_i are VBG_* on the set E for each $i \in \{1, \dots, n\}$.*
(2) *If there exists a base $e = (e_1, \dots, e_n)$ of the space X for which each mapping F_i, $i \in \{1, \dots, n\}$, is VBG_* on E then F is VBG_* on E.*

Theorem 1. [5] *Let $E \subset [a, b]$. If a function $F\colon [a, b] \to \mathbb{R}$ is VBG_* on the set E, then F is differentiable at a set of full Lebesgue measure.*

The obvious corollary of this theorem for a mappings which take values in a real normed space is as follows:

Corollary 1. *Let X be a real normed space, $\dim X < \infty$ and $E \subset [a, b]$. If a mapping $F\colon [a, b] \to X$ is VBG_* on the set E, then F is differentiable (in the Fréchet sense) at almost all points of this set.*

Definition 7. [6] *Let $\emptyset \neq M \subset Z$, where Z is a real normed space. Let z belong to the closure of the set M. The set*

$$\Big\{v \in Z\colon \exists (z_n)_{n\in\mathbb{N}}, z_n \in M, \lim_{n\to\infty} z_n = z, \exists (\lambda_n)_{n\in\mathbb{N}}, \lambda_n > 0\colon \lim_{n\to\infty} \lambda_n (z_n - z) = v\Big\}$$

is called the tangent cone to M at z and is denoted by $\operatorname{Tan}(M, z)$. The elements of $\operatorname{Tan}(M, z)$ are called vectors tangent to M at z. The set $\operatorname{Tan}(M, z)$ is also called the contingent of M at z (see [1], [5]*).*

The basic properties of the contingent and the connections between differentiability of a mapping $f\colon X \to Y$ at a point, where X, Y are real normed spaces and the contingent of its graph one can find in [3], [4], [6], [7].

Definition 8. *If X is a real normed space, then a mapping f is called an embedding if it is a homeomorphism of the interval (a, b) into X, where $f\big((a, b)\big)$ is equipped with the subspace topology. A subset Γ of the space X is called a curve if there is an embedding $f\colon (a, b) \to X$ such that $f\big((a, b)\big) = \Gamma$. This embedding is called a parametrization of the curve Γ.*

The following theorem gives a connection between the contingent of a curve and the existence of a differentiable parametrization of this curve.

Theorem 2. [8] *Let X be a real normed space for which $1 < \dim X < \infty$. Assume that for a curve $\Gamma \subset X$ the following conditions are fulfilled:*

(i) *for each $p \in \Gamma$ the contingent $\operatorname{Tan}(\Gamma, p)$ is one-dimensional linear subspace of X,*
(ii) *there exists a subspace Y of X such that $\operatorname{codim} Y = 1$ and*

$$\operatorname{Tan}(\Gamma, p) \not\subset Y$$

for each $p \in \Gamma$.

Then there exist an open interval (c, d) and a differentiable parametrization $g\colon (c, d) \to \Gamma$ of the curve Γ such that

$$\inf_{t \in (c,d)} \left\| g'(t) \right\| > 0.$$

Corollary 2. [8] *Let $f\colon (a, b) \to \Gamma$ be a parametrization of the curve Γ. Then under assumptions of theorem 2, for every open interval (c, d) there exists a mapping $g\colon (c, d) \to \Gamma$ such that the mapping $g^{-1} \circ f\colon (a, b) \to (c, d)$ is an increasing homeomorphism.*

Corollary 3. [8] *Under assumptions of theorem 2, each parametrization of the curve Γ is almost everywhere differentiable.*

Theorem 3. [2] *A mapping $F\colon [0, 1] \to \mathbb{R}$ is continuous and VBG_* on the interval $[0, 1]$ if and only if there exists a homeomorphism $h\colon [0, 1] \to [0, 1]$ such that $F \circ h$ is differentiable.*

We will use the following easy generalization of theorem 3.

Theorem 4. *Let $F\colon [0, 1] \to \mathbb{R}$. The mapping F is continuous and VBG_* on $[0, 1]$ if and only if there exists a homeomorphism $h\colon [c, d] \to [0, 1]$ such that $F \circ h$ is differentiable.*

2. Main results

Applying theorem 2., lemma 2. and theorem 4. we will prove that each parametrization of a curve Γ satisfying assumptions of theorem 2 is VBG_*. The following theorem is a partial converse of the corollary 1.

Theorem 5. *Let X be a real normed space such that $1 < \dim X < \infty$. Assume that for a curve $\Gamma \subset X$ the following conditions are fulfilled:*

- (i) *for each $p \in \Gamma$ the contingent $\mathrm{Tan}(\Gamma, p)$ is one-dimensional linear subspace of X,*
- (ii) *there exists a subspace Y of X such that $\mathrm{codim} Y = 1$ and*

$$\mathrm{Tan}(\Gamma, p) \not\subset Y$$

for each $p \in \Gamma$.

Then each parametrization $f\colon (a, b) \to \Gamma$ of the curve Γ is VBG_* in (a, b).

Proof. Let $f\colon (a, b) \to \Gamma$ be a parametrization of the curve Γ. By theorem 2, there exists a differentiable parametrization $g\colon (c, d) \to \Gamma$ of that curve. Obviously $f^{-1} \circ g$ is a homeomorphism of (c, d) onto (a, b).

Fix an interval $[c_1, d_1]$ contained in (c, d). Then there exists an interval $[a_1, b_1]$ in the set (a, b) such that the mapping

$$\left(f^{-1}\circ g\right)|_{[c_1,d_1]}\colon [c_1,d_1]\to[a_1,b_1]$$

is a homeomorphism of $[c_1,d_1]$ onto $[a_1,b_1]$.

Denote $h^*=\left(f^{-1}\circ g\right)|_{[c_1,d_1]}$, $f^*=f|_{[a_1,b_1]}$ and $g^*=g|_{[c_1,d_1]}$. Obviously, f^* is continuous and g^* is differentiable.

Fix a base $e=(e_1,\dots,e_n)$ of the space X. Then

$$f^*(t)=\sum_{i=1}^{n} f_i^*(t)e_i \quad\text{and}\quad g^*(\tau)=\sum_{i=1}^{n} g_i^*(\tau)e_i,$$

where $f_i^*\colon [a_1,b_1]\to\mathbb{R}$, $g_i^*\colon [c_1,d_1]\to\mathbb{R}$, $i\in\{1,\dots,n\}$ and $t\in[a_1,b_1]$, $\tau\in[c_1,d_1]$. Since $g^*=f^*\circ h^*$, then $g_i^*=f_i^*\circ h^*$ for each $i\in\{1,\dots,n\}$. The mapping g^* is differentiable, so g_i^* is differentiable if $i\in\{1,\dots,n\}$.

Moreover, h^* is a homeomorphism and f_i^* is continuous if $i\in\{1,\dots,n\}$ and by theorem 4 we have $f_i^*\in VBG_*\big([a_1,b_1]\big)$ for each $i\in\{1,\dots,n\}$.

By lemma 2(2) we conclude that the mapping

$$f^*\colon [a_1,b_1]\to X$$

is VBG_* in $[a_1,b_1]$. Therefore the mapping f is VBG_* on each closed subinterval of (a,b). The interval (a,b) is a countable union of closed subintervals, so the mapping f is VBG_* on (a,b). □

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AN ESTIMATION METHOD OF TRAFFIC SAFETY LEVEL OF VEHICLES ON A LEVEL CROSSING

ANDRZEJ YATSKO

Abstract

Let us consider a level crossing as a technical object with a certain level of safety. In the paper we propose a new method based on the theory of geometric programming. It allows to solve the problem of minimizing common risk of the object safety violation in a simple analytic form due to the choice of object protection parameter set. There is given a numerical example illustrating the calculated scheme of the method.

1. Introduction

It is known that a large proportion of traffic incidents is committed on level crossing. Thus a task of ensuring safety on level crossing is relevant. Here we consider a simplified formulation and solution of the task, since the total volume of its solution is a problem and it is beyond the scope of this study.

2. Main results

In this paper the traffic on the level crossing is considered as a complex object with a certain level of safety. Consider the following safety threats:
U_1 – drive over level crossing at red traffic light by drivers of a group I;
U_2 – drive over level crossing at red traffic light by drivers of a group II;
U_3 – a vehicles collision on the level crossing that does not stop on the tracks;
U_3 – collision with a train and other traffic incidents, which lead to a stop of transport on the tracks.

The group II consists of a car thieves, a drunk drivers, pursued criminals and other persons whose contact with the police is tantamount to arrest them. The group I consists of violators which do not belong to the group

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II. Common threat U of object safety consists of at least one of the threat U_1, U_2, U_3, U_4: $U = \bigcup U_i$.

In fact, the number of threats is much more than $n = 4$. But to keep things simple we'll assume $n = 4$.This is sufficient to illustrate the proposed method of estimating and minimizing the common risk of the object safety violation.

Suppose that the events U_i are independent, $i = 1, 2, 3, 4$, and the probability $y = P(U)$ (common risk of the object safety violation) is expressed as a sum of particular risks u_i :

$$y = u_1 + u_2 + u_3 + u_4, \tag{1}$$

where $u_1 = P(U_1), u_2 = P(U_2)(1 - u_1), u_3 = P(U_3)(1 - u_1)(1 - u_2), u_4 = P(U_4)(1 - u_1)(1 - u_2)(1 - u_3)$.

In addition,

$$u_1 = P(U_1) \approx \frac{M_1}{N_1}.$$

The fraction $\frac{M_1}{N_1}$ is an estimate of the particular risk u_1, where N_1 is total number of vehicles which passed through the crossing for time T (let us say $T = 1$ day) and M_1 is number of drivers of the group I which passed through the crossing for time T at red traffic lights.

Similarly, we estimate other particular risks, for example,

$$u_2 = P(U_2)(1 - u_1) \approx \frac{M_2}{N_2}.$$

The fraction $\frac{M_2}{N_2}$ is an estimate of the particular risk u_2, where N_2 is total number of vehicles which passed through the level crossing for time T without violators of the group I. Number M_2 is number of drivers of the group II which passed through the level crossing for time T at red traffic lights.

The object in question has safety protection system. This system includes signal operator, road inspectors, technical means of preventing violations such as barriers, remote control system barriers etc.

Protection system gives the following parameters:
x_1 – time of duty by road inspectors;
x_2 — time between duty;
x_3 – average time between the opening and closing of the barriers.

In fact, the number of parameters x_i is much more than $m = 3$. But this is enough to illustrate the estimation method of minimizing the common risk of the object safety violation.

A Table below is a fragment of empirical data for 10 observations, which we used for the calculations. Each row in the Table corresponds to time T=1 day for situation on the level crossing.

Probability of threats U_i (multiplied by 10^3)				Protection parameters (in hours)		
u_1	u_2	u_3	u_4	x_1	x_2	x_3
2.1	0.20	5.8	0.51	2	4	0.25
0.42	0.53	1.09	0.20	2.5	3	0.20
0.75	0.29	3.00	0.63	3	5	0.25
2.0	0.52	0.17	0.20	1.5	2	0.10
5.9	0.16	3.80	0.43	1.3	3	0.20
1.8	0.35	1.12	0.12	1.3	2	0.20
0.30	0.53	3.08	0.52	4	5	0.25
2.8	0.23	1.28	0.40	1.7	3	0.15
4.9	0.13	7.50	1.22	2	5	0.20
3.3	0.20	1.44	2.8	3	6	0.10

These data are mainly expert evaluation of road inspectors, ambulance workers and staff which services the technical means of preventing violations. Following our paper [1], we assume that the vector $x = (x_1, x_2, x_3)$ of protection parameters is positive.

Let $u_i = u_i(x)$ be a polynomial

$$u_i = u_i(x) = C_i \cdot \prod_{j=1}^{3} x_j^{a_{ij}}, \quad C_i > 0, \quad i = 1, 2, 3. \tag{2}$$

A matrix $A = (a_{ij})$ is called an exponent matrix.

Taking the logarithm of both side of (2), we obtain

$$\ln u_i(x) = a_{i0} + a_{i1} \ln x_1 + a_{i2} \ln x_2 + a_{i3} \ln x_3, \quad i = 1, 2, 3, 4, \quad C_i = e^{a_{i0}} \tag{3}$$

Thus, we can get coefficients a_{ij} by methods of linear regression analysis [2]. Writing the equation (3) for the first row of the Table, we obtain for the risk u_1:

$$\ln 2,1 = a_{10} + a_{11} \ln 2 + a_{12} \ln 4 + a_{13} \ln 0,25, \tag{4}$$

or

$$a_{10} + 0,693a_{11} + 1,386a_{12} - 1,386a_{13} = 0,742$$

Similarly, writing the equation (3) for next tows of the Table we obtain an algebraic system

$$Fa^1 = w^1, \tag{5}$$

where $a^1 = (a_{10} \quad a_{11} \quad a_{12} \quad a_{13})^T$ is a column of vector of required coefficients in the polynomial $u_1(x) = C_1 \cdot \prod_{j=1}^{3} x_j^{a_{1j}}$, $C_1 = e^{a_{10}}$ and the matrix F is expressed as $F = (\mathbf{1}, \ln x^1, \ln x^2, \ln x^3)$.

Moreover, $\mathbf{1}$ is a column of ones; $\ln x^j$ j=1,2,3, is a column for values $\ln x_{js}$ of the $\ln x_j$ for the parameter x_j and 10 observations $s = 1, 2, \ldots, 10$; w^1 is a column for values $\ln u_{is}(x)$ of the $\ln u_i(x)$;

$$w^1 = (\ln 2.1 \quad \ln 0.42 \ldots \quad \ln 3.3)^T.$$

According to the method of least squares (MLS) the solution $a^i = \dot{a}^i$ of the equation (5) is given in the form [2]:

$$\dot{a}^1 = F^+ w^1 \tag{6}$$

Here F^+ is so called pseudo-inverse matrix of the matrix F. Calculation method for the matrix F^+ is given in the paper [4]. Recall that the pseudo-inverse is defined and unique for all matrices whose entries are real or complex numbers. Vector $\dot{a}^i$ is a solution of equation (5) under the condition that the equation is compatibility. In the converse case, $\dot{a}^i$ is the best approximation solution (according to the MLS).

Thus,

$$\dot{a}^1 = F^+ w^1 = \begin{pmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \begin{pmatrix} -2.08 \\ -4 \\ 3 \\ -1 \end{pmatrix}, \quad C_1 = e^{-2.08} = 0,125$$

and the polynomial u_1 is expressed as

$$u_1 = u_1(x) = 0,125 x_1^{-4} x_2^3 x_3^{-1}$$

Using the given calculate scheme for the risks u_2, u_3, u_4, we obtain

$$u_2 = u_2(x) = 0,8 x_1^2 x_2^{-2},$$
$$u_3 = u_3(x) = 6 x_1^{-2} x_2^3 x_3^2, u_4 = u_4(x) = 0,004 x_1^{-1} x_2^3 x_3^{-1}.$$

Thus, the common risk y at the interval [0,T] of time (T equals 1 day) is expressed as

$$y = f(x) = 0,125 x_1^{-4} x_2^3 x_3^{-1} + 0,8 x_1^2 x_2^{-2} + 6 x_1^{-2} x_2^3 x_3^2 + 0,004 x_1^{-1} x_2^3 x_3^{-1}$$

Coefficient of variation $\dot{V}$ is used for precision and sufficiency to empirical data:

$$\dot{V} = \frac{\dot{\sigma}}{\dot{y}} 100\%$$

Here

$$\dot{y} = \dot{u_1} + \dot{u_2} + \dot{u_3} + \dot{u_4}$$

Moreover $\dot{u}_i$, i=1,2,3,4, is a sample mean of observations u_{is}, s=1,2,...10; $\dot{\sigma}^2$ is the sum of the sample variance $\dot{\sigma_i}^2$:

$$\dot{\sigma_i}^2 = \| F\dot{a}^i - w^i \|^2 /(N-m-1),$$

where N is the total number of observations, but m is the number of protection parameters (m=3); $\|\cdot\|$ means an euclidean norm of vector; $\dot{\sigma_i}^2$ is the sample variance according to the MLS-solutions

$$\dot{a}^i = F^+ w^i$$

for the algebraic system

$$Fa^i = w^i, \quad i = 1,2,3,4.$$

In our case,

$$\dot{\sigma} = \sqrt{\dot{\sigma_1}^2 + \dot{\sigma_2}^2 + \dot{\sigma_3}^2 + \dot{\sigma_4}^2},$$

We obtained $\dot{V} = 9\%$ as a result of data processing for N=100. This result gives acceptable discrepancy between the experimental and calculated data [5].

It should be found the vector $x = x_* > 0$, with components x_{j^*} such that value $y_* = f(x_*)$ is minimal. Using exponents a_{ij}, we can write the exponent matrix A as

$$A = (a_{ij}) = \begin{pmatrix} B \\ H \end{pmatrix} = \begin{pmatrix} -4 & 3 & -1 \\ 2 & -2 & 0 \\ -2 & 3 & 2 \\ -1 & 3 & -1 \end{pmatrix},$$

where sub-matrices

$$B = \begin{pmatrix} -4 & 3 & -1 \\ 2 & -2 & 0 \\ -2 & 3 & 2 \end{pmatrix}, \quad H = (-1 \quad 3 \quad -1).$$

Note that $detB \neq 0$. It follows that exist an inverse matrix B^{-1} :

$$B^{-1} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} = \begin{pmatrix} -2 & -\frac{9}{2} & -1 \\ -2 & -5 & -1 \\ 1 & 3 & 1 \end{pmatrix}.$$

In our case sub-matrix $H = (-1 \quad 3 \quad -1)$ contains one row of matrix A, which do not belong to the sub-matrix B. Hence the example has the first difficulty level (see [1]).

Using the formulas from [1], we get subsidiary variables δ_i. These ones are called dual variables and are found by the formula

$$\delta^T = (\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4) = \frac{1}{\mu}\left(-H \cdot B^{-1}, 1\right) = \frac{1}{\mu}\begin{pmatrix} 5 & \frac{27}{2} & 3 & 1 \end{pmatrix},$$

where the number

$$\mu = 5 + \frac{27}{2} + 3 + 1 = \frac{45}{2}$$

Therefore

$$\delta^T = (\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4) = \frac{1}{45}(10 \quad 27 \quad 6 \quad 2).$$

Thus,

$$\delta_1 = \frac{10}{45}, \quad \delta_2 = \frac{27}{45}, \quad \delta_3 = \frac{6}{45}, \quad \delta_4 = \frac{2}{45}.$$

Using the formulas of the paper [3], we can write the minimal value y_* multiplied by 10^3 of the common risk y due to

$$C_1 = 0.125,\ C_2 = 0.8,\ C_3 = 6.0,\ C_4 = 0.004.$$

In our case

$$(7) \quad y_* = \prod_{i=1}^{4} \left(\frac{C_i}{\delta_i}\right)^{\delta_i} = \left(\frac{0.125}{10}\right)^{\frac{10}{45}} \left(\frac{0.8}{27}\right)^{\frac{27}{45}} \left(\frac{6}{6}\right)^{\frac{6}{45}} \left(\frac{0.004}{2}\right)^{\frac{2}{45}} = 1.55,$$

i.e. minimal value of the common risk is 0,155%.

Then the protection parameters x_{j^*} , j=1,2,3, can be found from the relationships

$$x_{1*} = \prod_{i=1}^{3} \left(\frac{\delta_i \cdot y_*}{C_i}\right)^{k_{1i}} = \left(\frac{10 \cdot 1.55}{45 \cdot 0.125}\right)^{-2} \left(\frac{27 \cdot 1.55}{45 \cdot 0.8}\right)^{-\frac{9}{2}} \left(\frac{6 \cdot 1.55}{45 \cdot 6}\right)^{-1},$$

$$x_{2*} = \prod_{i=1}^{3} \left(\frac{\delta_i \cdot y_*}{C_i}\right)^{k_{2i}} = \left(\frac{10 \cdot 1.55}{45 \cdot 0.125}\right)^{-2} \left(\frac{27 \cdot 1.55}{45 \cdot 0.8}\right)^{-5} \left(\frac{6 \cdot 1.55}{45 \cdot 6}\right)^{-1},$$

$$x_{3*} = \prod_{i=1}^{3} \left(\frac{\delta_i \cdot y_*}{C_i}\right)^{k_{3i}} = \left(\frac{10 \cdot 1.55}{45 \cdot 0.125}\right) \left(\frac{27 \cdot 1.55}{45 \cdot 0.8}\right)^{3} \left(\frac{6 \cdot 1.55}{45 \cdot 6}\right),$$

Thus, we get optimal protection parameters (in hours)

$$x_{1^*} = 1.98, \quad x_{2^*} = 1.84, \quad x_{3^*} = 0.15$$

3. Final remarks

Thus, if the proposed model is acceptable with respect to the coefficient of variation, it allows to solve the problem of minimizing common risk of the object safety violation in a simple analytic form due to the choice of object protection parameter set.

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EYES, OPTICS AND IMAGING: MATHEMATICS AND ENGINEERING INNOVATIONS INSPIRED BY NATURE

RASTISLAV TELGÁRSKY

Abstract

This report is about mathematics and engineering innovations inspired by the nature, in particular, about the optical devices and imaging technologies inspired by eyes of living organisms. Instead of reviewing existing mathematical methods, the author explains visual sensors in nature and also the corresponding engineering devices for further inspiration of mathematicians. *Contents:* Introduction. Eyes. Optics and Imaging. Camouflage. References. Figure Credits.

1. Introduction

The effort of mathematicians in 21st century can be divided into mathematics applied in engineering and sciences, and into abstract mathematics, which has no immediate connection to any application. The research in biology brings amazing discoveries about complexity and functioning of living organisms from viruses, bacteria, and simple organisms to complex bodies and functions of mammals. In all new discoveries the nature offers problem-solving ideas inspiring human creativity, in particular, in engineering and new technologies for vision sensors. Thus this paper focuses on eyes, optics, and imaging, which inspires mathematical modelling and engineering.

There are hundreds of books about the topics of this paper. Here are just few I have selected as complementary to this paper: Adam [1], Barrow [2], Dawkins [3], Henderson [4], Nowak [5], Tegmark [6], and Yahya [7], [8].

2. Eyes

Earthworms have no eyes, but they do possess light sensitive organs – the receptor cells. The cyclops, also called waterflies, have only one eye. Fish, birds, mammals, and many other species have two eyes. Each eye has lens

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with variable focusing, retina and many neural connections to the brain. Some of these species have eyes on sides of the head so that they get large spherical angle of vision, while others have both eyes in front of the face to provide the stereo vision with field depth, and good estimation of sizes and distances. Notostracans Triops (tadpole shrimps) have 3 eyes: 2 compound eyes plus the larval nauplius eye.

Honey bees and bumble bees have 5 eyes: 2 compound eyes, each composed of 4500 "little eyes" called ommatidia (Wikipedia: Ommatidium), and 3 simple eyes, called ocelli (Wikipedia: Simple eye in invertebrates) on top of the head (Figure 1). Honey bees can see in ultraviolet light. Their vision is trichromatic like ours, but their visible spectrum is shifted toward the ultraviolet (westmtnapiary.com/Bees_and_color.html). The sand spider

(A) Male Carpenter Bee (B) Honey Bee Face

FIGURE 1. 5 eyes of bees: 2 compound and 3 ocelli

Sicarius Hahni has 6 eyes. The *jumping spiders*, and many other spiders, have 8 eyes (Figure 2). The eye locations have various configurations, but typically 4 eyes are on the face and form very sharp images, while the other 4 eyes are mostly motion detectors. The *starfish* has 5, 21 or 40 arms (legs), and has an eye at the end of each arm (Figure 3). Each eye is sensitive to light, but does not form an image. A single *scallop* (Figure 4) can possess over a hundred eyes! Each eye has a lens and a retina which is attached to a branch of the optic nerve (Figure 5). The *giant clam* of the South Pacific, Tridachna gigas, has over 1000 eyes, but these eyes are relatively simple *photoreceptors*.

The *housefly*, Musca domestica, has two *compound eyes* (Figure 6), called ommatidia, capable of tracking movements up to five times faster than our own eyes!

(A) Hogna Wolf Spider (B) Jumping Spider

FIGURE 2. Eye arrangements of 8-eye spiders

(A) Starfish with 11 arms (B) Crown of Thorns

FIGURE 3. Starfish

(A) Full view of a scallop (B) Close-up with eyes

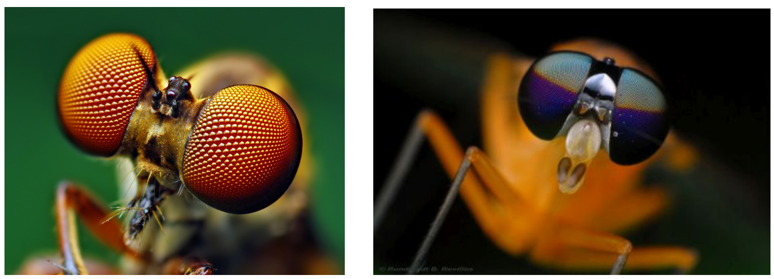
FIGURE 4. Scallop with hundreds of eyes

The *fruit flies* have very small compound apposition eyes that require neural processing. They can sense rapid motion approaching 200 cycles per second. Considerable image processing is confined to a very small space.

The fish called *Stoplight Loosejaw* (Malacosteus niger) has two *bioluminescence* spots, one emitting a low frequency red light, and the other bright

(A) Few blue eyes (B) Single blue eye

FIGURE 5. Close-up to scallop eyes



(A) Robber fly (B) 360°view

FIGURE 6. Compound eyes of flies

green or yellow light (Figure 7). The illumination of its surrounding with the red light allows it to see around but be invisible, because other creatures cannot detect this light. When close to a prey, the fish lights up the bright

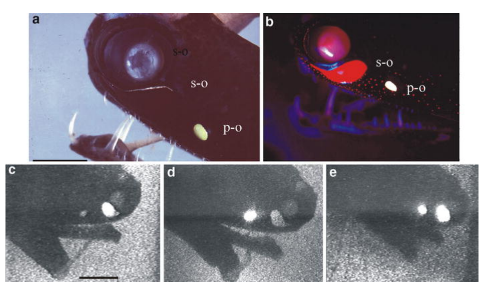


FIGURE 7. Stoplight Loosejaw uses bioluminescence from 2 spots

bioluminescence spot to see the details before catching the prey. Another feature is the loose jaw, that is, the low jaw without the bottom allowing fast closing the mouth with sharp teeth.

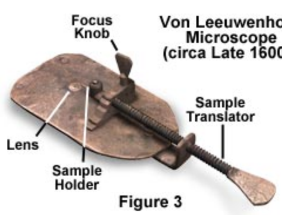
3. Optics and Imaging

A drop of water on a leaf works like a natural magnifying lens, but is has a considerable distortion (Figure 8).

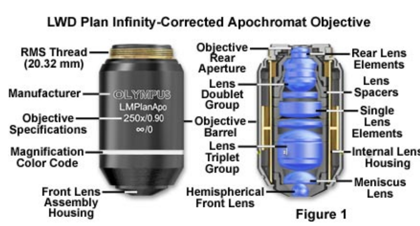
Figure 8. Water drops on a leaf

The lenses cut precisely from a glass have more desirable properties. The purpose of lenses is the magnification of images of objects, with an exception of a door viewer, which minifies the images, and provides a much wider angle of view.

The *microscope* was invented by Antonie van Leeuwenhoek (Figure 9A), who immediately discovered a gallery of small living organisms. The single lens was later replaced by an array of lenses. An Olympus objective for a microscope is shown in Figure 9B for a comparison.



(A) Leeuwenhoek device



(B) Olympus objective

Figure 9. Comparison of microscope's lenses

The research in chemistry, geology, biology and medicine is unthinkable without microscopes. Here is a modern microscope, shown in Figure 10, connected to a sensitive digital camera and a large digital monitor.

FIGURE 10. Modern microscope with camera and monitor

Another breakthrough in science was the invention of *telescope* by Hans Lipperhey, a German spectacle maker, who combined curved lenses to magnify objects by up to 3 times. He received the patent in 1608. Galileo Galilei heard news of the telescope, and constructed his own version of it without ever seeing one. Instead of the initial 3 power magnification, he crafted a series of lenses that in combination allowed him to magnify things by 8, 20 and eventually 30 times (Figure 11). He soon discovered the rings of Saturn.

FIGURE 11. Galileo's telescope

From telescopes for visible lights the development went to infrared, ultraviolet, X-ray and radio waves telescopes. The infrared light better penetrates the earth atmosphere, but still, the earth atmosphere acts like liquid lenses and distorts the images from space. Therefore a technique was developed to detect and correct the shifts of phases in image elements. This is called the *Adaptive Optics System* (Figure 12). The result of the phase

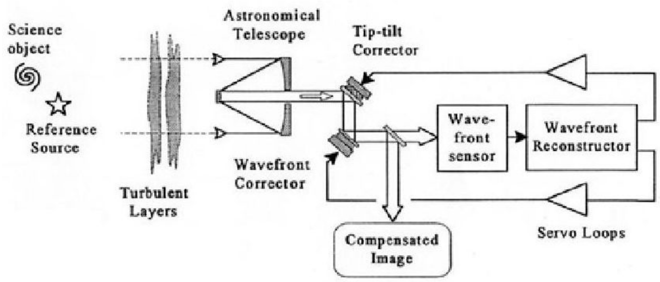


FIGURE 12. Astronomical Adaptive Optics System

correction is seen in the infrared image of Neptune (Figure 13).

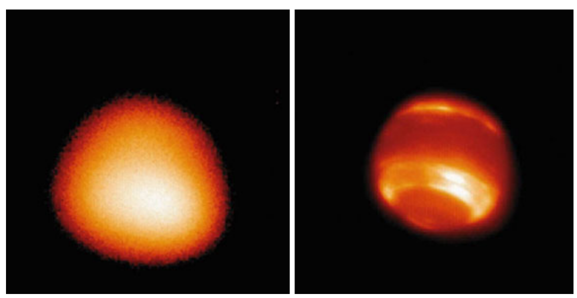


FIGURE 13. Infrared image of Neptune – without and with AO

Another improvement of image quality is achieved by placing telescopes on tops of high mountains, or to the orbit of the earth, like the *Hubble Space Telescope* (Figure 14A), to the orbit of the sun, or even farther in cosmic space. However, for many astronomy hobbyists and enthusiasts, the Meade telescope ETX70A with 882 tripod and software (Figure 14B) is of very high quality, and even it has the star/planet tracking system, called *sidereal time*. So, a large number of people have astronomical observatories in back yard or front yard of their houses, or they take trips out of town where the air is cleaner for the exploration of the night sky.

The energy of electromagnetic waves coming from the space, except from the sun, is very small. So, large arrays of telescopes or radiotelescopes are built and connected, so that the compound image is constructed by computer algorithms. With larger energy per received bandwidth, the algorithms can even calculate the spectra of incoming light or radio waves. Analysis of these spectra allows for remote chemical analysis of the sources.

(A) Hubble Space Telescope

(B) Meade Telescope

FIGURE 14. Large and small telescopes

In particular, there is a fascinating possibility of discovering a life on an earth-like planet to a sun-like star somewhere far from our solar system, or at least, the physical and chemical conditions that may support the life forms.

In much lower frequencies, which propagate through the air, *bats* (Figure 15A) generate the *ultrasound* and reconstruct the range-Doppler images from its reflections. This well supplements their vision, and enables them to navigate and catch insects at night (Figure 15B).

(A) Vampire bat

(B) Bat eating an insect

FIGURE 15. Bats

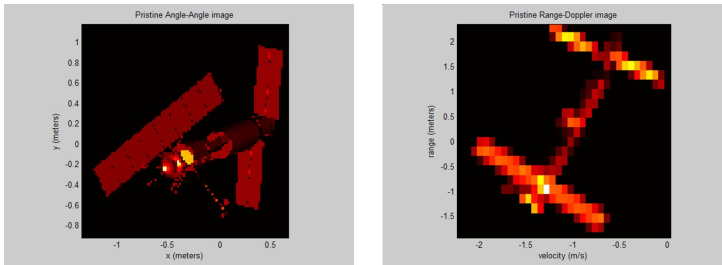
The *ultrasound imaging systems* used in the medical diagnostic devices (Wikipedia: Medical ultrasonography) are very sophisticated extensions of bat's image processing, although the principle is the same: the ultrasound pulses are sent, reflected, received, and processed into images. The system consists of a cylindrical hand-held scanning device containing the ultrasound transmitter and receiver, which is connected to the computer processing unit with a real-time display and a real-time recording system. The most important uses of the ultrasound medical system are the diagnoses of the heart (Figure 16A), and monitoring the baby and the expecting mother health during the pregnancy (Figure 16B). However, it is also used in diagnostics of kidneys, gallbladder, bladder, and other organs. There is a hand-held version of the entire image processing unit, which is not much larger than a smart phone (Figure 16C).

(A) Heart Valves (B) Pregnancy (C) Hand-held device

FIGURE 16. Medical ultrasonography

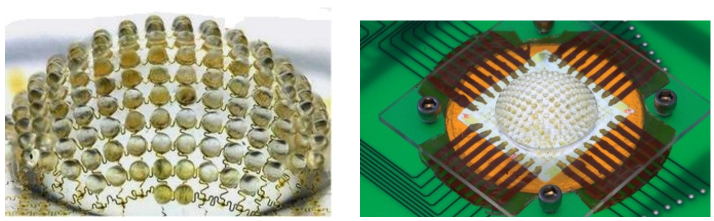
The *radar imaging system* works similarly like the bat's ultrasound system, however, the radar can scan (sweep) a larger area (a spatial angle), and thus construct image of a very large scene. The invention of device called *laser*, which produces a narrow beam of light or infrared light pulses, enabled the construction of an imaging system called *ladar*. Ladar uses the short-time Fourier transform, also called the time-frequency transform, to process the reflected pulses into the range-Doppler image, which is a 2D phase space, where the horizontal coordinate is the radial distance and the vertical coordinate is the radial velocity of the illuminated object. The standard optical image of a satellite is in Figure 17A, and the corresponding range-Doppler image is in Figure 17B.

The compound eyes of insects are faster than eyes with lens and retina. U.S. Navy is studying fly compound eyes to help develop guidance systems for weapons, and more compact optical sensors (Figure 18). Hopefully, the intelligent vision processing will provide new technology for robotics, and other industrial or military applications. The cameras with compound lenses (Figure 18A) imitate the compound eyes of insects. The 100 megapixel camera (Figure 18B) is capable of producing images with $10,240 \times 10,240$ pixels.



(A) Angle-Angle Image (B) Range-Doppler Image

FIGURE 17. Images of a Satellite



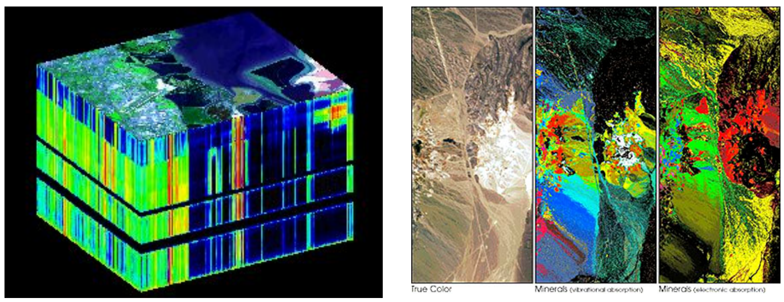
(A) Compound lenses (B) Wide-angle lenses

FIGURE 18. Cameras inspired by compound eyes of bugs

Stereo vision allows for depth perception and estimation of distances. However, the laser illumination of target provides an active and more precise estimation of the distance. The computer vision algorithms take a lot from eyes and compound eyes for processing the digital images. Panoramic images are created by stitching together shifted images to cover a larger view angle. The integration of images from multiple cameras is used in airplane cockpits. The planned Amazon's smartphone should have the total of 6 cameras, where 4 cameras will sense the hand gestures without the user touching the screen. Robots with vision sensors can perform complex functions such as a product assembly or reaching hard places. Hemispherical vision systems had been developed for monitoring and management of airports/airfields. This technology can be replaced by the multiple cameras modelling the compound eye of a honey bee. Multi-spectral vision systems put into one display the patches of images and pseudocolour images from different wavelengths, for example, from infrared or other invisible wavelengths.

Hyperspectral vision systems split the visible spectrum into disjoint intervals, say, 100 intervals, and calculate the pixel intensities for each interval.

The resulting image is actually a stack of 100 images, or one 3D image, called datacube (Figure 19A). Each bandwidth, corresponding to an interval of the spectrum, has a specific sensitivity to the chemical structures of regions in the field of view. Note that there are two layers in the datacube without data. These are caused by atmospheric absorption bands, specifically, H_2O at 1.5 and 2.0 microns. Many useful properties can be detected when looking at forests, fields, marshes, valleys, etc. (Figure 19B).



(A) HS Datacube (B) Aviris Minerals

Figure 19. Remote Sensing Images

The mantis shrimp (Wikipedia: Mantis shrimp) has one of the most elaborate visual systems ever discovered in animal kingdom. Each of two eyes carry 16 photoreceptor pigments which allow to perceive both polarized light and multispectral images. Moreover, each eye possesses trinocular vision and depth perception.

4. Camouflage

The *camouflage* used by insects, sea creatures and other species appears to protect them from predators, and shows that their evolution and brain are surprisingly advanced. The camouflage consists in mimicking the background, and this function is either permanent or adaptive. A moth or a beetle have very similar patterns to tree barks, where they rest (Figure 20).

(A) Pine Hawk Moth (B) Monochamus Notatus

FIGURE 20. Moth and beetle camouflaged on tree bark

The Leaf Insect resembles leaves of plants in its environment (Figure 21A). Similarly, the Leafy Sea Dragon has shapes almost identical to leaves in its sea environment (Figure 21B).

(A) Leaf insect (B) Leafy sea dragon

FIGURE 21. Insect and sea dragon look like leaves

The adaptive camouflage depends on living environment. A fish can create a cover-up by mimicking the color patterns of an octopus (Figure 22).

FIGURE 22. Fish and octopus

The brain of this fish performs very complex image processing algorithms to change colors of its skin according to what it saw before assuming the camouflaged position.

This fish behaviour inspires some researchers to work on invisibility devices. First is the kind of cloth, which becomes invisible, when the surface, turned to the observer, maps the background behind the person wearing the cloth. To the observer it looks like there is nobody there because the seen background is not interrupted by the spot occupied by the person. Except the invisibility of objects in visible light, there are also studied surfaces and materials making objects invisible to radars, such as stealth planes, stealth helicopters and stealth drones.

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Part III
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PROFESOR TADEUSZ ŚWIĄTKOWSKI – DOBRY DUCH POLITECHNIKI ŁÓDZKIEJ

IZABELA JÓŹWIK, MAŁGORZATA TEREPETA

Abstract

Profesor Tadeusz Świątkowski był wieloletnim pracownikiem i dyrektorem Instytutu Matematyki Politechniki Łódzkiej. Prezentujemy jego sylwetkę jako wielkiego matematyka i wspaniałego nauczyciela oraz jego wkład w rozwój teorii funkcji rzeczywistych.

1. Wstęp

W tym roku mija 20 lat od śmierci naszego Profesora – Tadeusza Świątkowskiego. Obie byłyśmy Jego studentkami. Niestety bardzo krótko, ze względu na Jego problemy zdrowotne. Mimo upływu lat, pamiętamy jak jasno tłumaczył nawet skomplikowane zagadnienia matematyczne. Zawsze zależało Mu na tym, by pokazywać piękno i prostotę matematyki. Nigdy swoim zachowaniem nie mówił „patrzcie, jaki ja jestem mądry", tylko zawsze „patrzcie, jakie to jest proste". Do tej pory Profesor kojarzy nam się właśnie z tą cechą. Widać ją wyraźnie, gdy czyta się te Jego artykuły, które były pisane po polsku. Profesor chętnie pisał w ojczystym języku i publikował w lokalnych czasopismach. Dużą część jego prac można znaleźć w Zeszytach Naukowych Politechniki Łódzkiej i Zeszytach Naukowych Uniwersytetu Łódzkiego. Niestety, ich zasięg był przez to dość ograniczony, ale Profesor uważał, że należy wspierać swoje środowisko naukowe. Mimo to, wiele z jego artykułów znalazło swój oddźwięk w matematyce światowej i do tej pory Jego artykuły pojawiają się w bibliografii prac wielu znanych matematyków. Profesor był bardzo szczodry w dzieleniu się swoją wiedzą. Chętnie podsuwał problemy do rozważania, pomysły prac, a nawet szkice rozwiązań. Wypromował ok. 40 doktorów nauk matematycznych. Niestety,

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ze względu na reorganizację niektórych instytucji, udało nam się dotrzeć tylko do 24 nazwisk; lista doktorantów dostępna jest na stronie:

`http://genealogy.math.ndsu.nodak.edu/id.php?id=172152`

W tym artkule chciałybyśmy krótko przypomnieć Jego postać i te wyniki, które uważamy za istotne w teorii funkcji rzeczywistych, gdyż stały się podstawą do stworzenia takich pojęć, jak np. path derivatives, local systems, Świątkowski property, Świątkowski function.

2. Życiorys

Tadeusz Świątkowski urodził się 28 listopada 1933 roku w Bełchatowie. Tam rozpoczął swoją edukację. Po ukończeniu szkoły podstawowej uczęszczał do Liceum Ogólnokształcącego im. Bolesława Chrobrego w Piotrkowie Trybunalskim.

W 1951 roku rozpoczął studia matematyczne na Uniwersytecie Łódzkim. Rok przed ich ukończeniem w 1955 roku, rozpoczął pracę jako asystent na Politechnice Łódzkiej. W latach 1959-1962 pracował na Uniwersytecie Łódzkim, następnie ponownie na Politechnice Łódzkiej. W 1960 roku uzyskał stopień naukowy doktora nauk matematycznych, a rok później zdobył nagrodę PTM dla młodych matematyków. W 1962 roku wziął ślub z Marią Waliszewską (siostrą innego znanego łódzkiego matematyka – profesora Włodzimierza Waliszewskiego). Jego syn Stefan ukończył studia matematyczne, córka Joanna studia medyczne.

W 1966 uzyskał stopień naukowy doktora habilitowanego za pracę *O warunkach wystarczających monotoniczności funkcji.* Praca ta stanowiła pozytywne rozwiązanie problemu Z. Zahorskiego. Analogiczne rezultaty uzyskał (na całkowicie innej drodze) znany matematyk amerykański, autor wielu książek z teorii funkcji rzeczywistych, Andrew Bruckner ([1]).

W latach 1968-1972 współprowadził wykłady telewizyjne dla studentów studiów dla pracujących wyższych uczelni technicznych (program nosił tytuł „Politechnika TV"). Kierownikiem kursu był prof. W. Krysicki. Zespół wykładowców: prof. I. Dziubiński, prof. D. Sadowska, prof. T. Świątkowski, prof. L. Włodarski, był ściśle związany z Politechniką Łódzką. Tadeusz Świątkowski był współautorem popularyzatorskiej książki z geometrii (Geometria analityczna na płaszczyźnie i w przestrzeni, Włodzimierz Krysicki, Tadeusz Świątkowski, Jan Zydler), zbioru zadań dla kandydatów na wyższe uczelnie techniczne oraz poradnika matematycznego (*Poradnik matematyczny*: praca zbiorowa pod red. I. Dziubińskiego i T. Świątkowskiego).

W 1978 roku uzyskał tytuł profesora nadzwyczajnego. Był wieloletnim dyrektorem Instytutu Matematyki Politechniki Łódzkiej. Posiadał wiele odznaczeń między innymi Medal Komisji Edukacji Narodowej, Krzyż Kawalerski Orderu Odrodzenia Polski.

Zmarł 30 października 1994 roku.

3. Zainteresowania naukowe

Dorobek naukowy Profesora zawiera około 30 prac. Najwcześniejsze prace prof. Tadeusza Świątkowskiego dotyczą funkcji analitycznych, następnie jego zainteresowania skierowały się ku teorii funkcji rzeczywistych i jej związkom z topologią. Badał uogólnienia pojęcia granicy i pochodnej oraz klasy funkcji ciągłych.

W 1960 roku Tadeusz Świątkowski napisał pracę doktorską, składającą się z dwóch części:

(1) O funkcjach holomorficznych w kole jednostkowym, przyjmujących każda swoją wartość nieskończenie wiele razy.
(2) O przekształcaniu funkcji mierzalnej na całkowalną.

Promotorem był profesor Witold Janowski.

Wyniki zostały przedstawione w pracach: *Sur les fonctions holomorphes qui prennent dans le cercle unité une infinité de fois chacune de leurs valeurs* ([9]) oraz *Sur une transformation d'une fonction mesurable en une fonction sommable* ([10]).

W pierwszej części pracy doktorskiej Świątkowski przedstawił wynik dotyczący funkcji holomorficznych w kole jednostkowym: wykazał, że zbiór funkcji holomorficznych przyjmujących każdą swoją wartość nieskończenie wiele razy jest rezydualny w przestrzeni wszystkich funkcji holomorficznych.

W drugiej części pracy Tadeusz Świątkowski rozwiązał problem Czesława Ryll-Nardzewskiego sformułowany następująco:

Czy dla każdej funkcji f prawie wszędzie skończonej i mierzalnej w $\langle a, b\rangle$ istnieje homeomorfizm ϕ przedziału $\langle a, b\rangle$ na siebie, spełniający warunek

$$\int_a^b f\left(\phi(t)\right) dt < \infty?$$

Czy istnieje homeomorfizm taki, który przekształca funkcje mierzalne wielu zmiennych na funkcje całkowalne w sensie Lebesgue'a?

Odpowiedź na powyższe pytania jest pozytywna. Tadeusz Świątkowski udowodnił twierdzenie:

Twierdzenie. *Niech dana będzie funkcja $f(x_1, x_2, \ldots, x_n)$ prawie wszędzie skończona i mierzalna na kostce*

$$a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, n.$$

Wówczas istnieją funkcje $\phi_1(x_1)$, $\phi_2(x_1, x_2)$, $\ldots, \phi_n(x_1, x_2, \ldots, x_n)$ *takie, że*

$$a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, n$$

oraz

(1) $\phi_1(x_1)$ *jest ściśle rosnąca oraz* $\phi_1(a_1) = a_1$, $\phi_1(b_1) = b_1$,
(2) *dla ustalonych* $x_1, \ldots, x_{k-1}$ *funkcja* $\phi_k(x_1, x_2, \ldots, x_{k-1}, x_k)$ *jest rosnącą funkcją zmiennej* x_k *taką, że* $\phi_k(x_1, x_2, \ldots, x_{k-1}, a_k) = a_k$, $\phi_k(x_1, x_2, \ldots, x_{k-1}, b_k) = b_k$, $k = 2, \ldots, n$,
(3) *dla dowolnej dodatniej liczby* α

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\phi_1(x_1), \phi_2(x_1, x_2), \ldots, \phi_n(x_1, x_2, \ldots, x_n))|^\alpha \, dx_1 dx_2 \ldots dx_n < \infty.$$

Kilka lat później (1973, [13]) profesor Świątkowski wrócił do przedstawionego w pracy doktorskiej problemu i wraz z profesorem Władysławem Wilczyńskim udowodnił następujące twierdzenia:

Twierdzenie. *Jeżeli* $f \geq 0$ *jest funkcją półciągłą z dołu na* $[0, 1]$ *taką, że*

$$\int_0^1 f(x) dx = +\infty,$$

to zbiór tych $h \in H$ *(*H *oznacza zbiór homeomorfizmów* $h \colon [0, 1] \to [0, 1]$ *spełniających warunki* $h(0) = 0$, $h(1) = 1$*), dla których*

$$\int_0^1 f(h(x)) \, dx < +\infty,$$

jest pierwszej kategorii w H.

Twierdzenie. *Dla dowolnego ciągu* $\{f_n\}$ *funkcji mierzalnych, określonych i prawie wszędzie skończonych na* $[0, 1]$ *istnieje homeomorfizm* h *tego przedziału na siebie taki, że*

$$\int_0^1 |f_n(h(x))| dx < +\infty$$

dla dowolnego $n \in \mathbb{N}$.

Następnym bardzo istotnym wynikiem uzyskanym przez profesora Świątkowskiego było sformułowanie twierdzenia o warunkach wystarczających monotoniczności funkcji. Jednym z bardziej znanych twierdzeń tego typu, jest to, które studenci pierwszego roku studiów poznają podczas kursu analizy matematycznej: *Jeżeli f jest różniczkowalna w (a, b) oraz jej pochodna jest dodatnia w (a, b), to f jest funkcją rosnącą w tym przedziale.* Twierdzenie to było w różny sposób uogólniane: np. założenie o różniczkowalności funkcji było zastępowane przez pojęcie różniczkowalności funkcji w innym sensie, zbiór, w którym pochodna miała istnieć lub w którym była nieujemna, był mniejszy niż (a, b). Jedno z takich twierdzeń zostało sformułowane w 1928 roku przez G. Gołdowskiego oraz niezależnie, w 1930 roku, przez L. Tonellego:

Twierdzenie ([3], [16])**.** *Niech f będzie funkcją spełniającą w przedziale (a, b) warunki:*

(a1) *f jest ciągła,*
(b1) *f' (skończona lub nieskończona) istnieje wszędzie poza przeliczalną liczbą punktów,*
(c1) *$f'(x) \geq 0$ prawie wszędzie.*

Wówczas f jest funkcją niemalejącą na (a, b).

W 1939 roku twierdzenie to zostało uogólnione przez G. Tołstowa ([15]), a w 1950 roku przez Z. Zahorskiego ([17]).

Twierdzenie (Tołstow)**.** *Niech f będzie funkcją spełniającą w przedziale (a, b) warunki:*

(a2) *f jest aproksymatywnie ciągła,*
(b2) *f'_{ap} (skończona lub nieskończona) istnieje wszędzie poza przeliczalną liczbą punktów,*
(c2) *$f'_{ap}(x) \geq 0$ prawie wszędzie.*

Wówczas f jest funkcją ciągłą i niemalejącą na (a, b).

Twierdzenie (Zahorski)**.** *Niech f będzie funkcją spełniającą w przedziale (a, b) warunki:*

(a3) *f jest funkcją Darboux,*
(b3) *f' (skończona lub nieskończona) istnieje wszędzie poza przeliczalną liczbą punktów,*
(c3) *$f'(x) \geq 0$ prawie wszędzie na (a, b).*

Wówczas f jest funkcją ciągłą i niemalejącą na (a, b).

Rodziny funkcji, które spełniają założenia obu ostatnich twierdzeń nie są rozłączne, ani nie zawierają się w sobie nawzajem. Twierdzenie Zahorskiego jest równocześnie silniejsze niż twierdzenie Tołstowa (Zahorski zakładał własność Darboux zamiast aproksymatywnej ciągłości) i słabsze od niego (zakładał istnienie zwykłej pochodnej zamiast pochodnej aproksymatywnej). Poszukiwania uogólnień tych twierdzeń szły w różnych kierunkach, np. próbowano wykorzystać ich słabsze założenia: (a3) oraz (b2) i (c2). Niestety z tych założeń nie wynika fakt monotoniczności funkcji (przykład takiej funkcji można znaleźć w [1]). Ponieważ każda funkcja aproksymatywnie ciągła jest funkcją Darboux pierwszej klasy Baire'a, Zahorski sformułował następujące pytanie:

Czy każda funkcja Darboux pierwszej klasy Baire'a spełniająca założenia $(b2)$ *i* $(c2)$ *jest ciągła i monotoniczna?*

Pozytywną odpowiedź na to pytanie uzyskał Tadeusz Świątkowski i przedstawił w 1966 roku w swojej pracy habilitacyjnej *O warunkach wystarczających monotoniczności funkcji* ([11]).

Niech $\mathcal{J}$ oznacza klasę funkcji f spełniających warunki:

(1) każdy zbiór postaci $\{x\colon f(x) > M\}$ oraz $\{x\colon f(x) < m\}$ jest zbiorem typu F_σ i składa się tylko z obustronnych punktów kondensacji funkcji f

(2) jeżeli $f \in \mathcal{J}$, g jest ciągła, to $f + g \in \mathcal{J}$.

Oczywiście funkcje Darboux pierwszej klasy Baire'a należą do klasy $\mathcal{J}$.

Twierdzenie ([11])**.** *Niech f będzie funkcją spełniającą w przedziale (a, b) warunki:*

(A) $f \in \mathcal{J}$,

(B) f'_{ap} *(skończona lub nieskończona) istnieje wszędzie poza przeliczalną liczbą punktów,*

(C) $f'_{ap}(x) \geq 0$ *prawie wszędzie.*

Wówczas f jest funkcją niemalejącą i ciągłą w (a, b).

Równolegle, podobne rezultaty niezależnie od Świątkowskiego uzyskał matematyk amerykański A. M. Bruckner ([1]). Droga uzyskania tego wyniku była jednak całkowicie inna.

Następny okres pracy naukowej Profesora poświęcony był uogólnieniu pojęcia granicy i pochodnej. W niektórych zagadnieniach pochodna zwyczajna może być zastąpiona przez pewne uogólnienia pochodnej. Z reguły odbywa się to poprzez zastąpienie zwykłej granicy ilorazu różnicowego przez inne przejście graniczne, np. przez wprowadzenie mocniejszej topologii lub ograniczenie się do punktów pewnego zbioru. W pracy [12] profesor Świątkowski wprowadził następujące pojęcie T-granicy i T-pochodnej.

Niech z każdym punktem x pewnego przedziału (a, b) będzie związana pewna rodzina $T(x)$ podzbiorów tego przedziału taka, że:

(a) *jeżeli* $E_1, E_2 \in T(x)$, *to* $E_1 \cap E_2 \in T(x)$,

(b) *jeżeli* $\delta > 0$ *to* $(x - \delta, x + \delta) \cap (a, b) \in T(x)$,

(c) x *jest jedynym punktem wspólnym rodziny* $T(x)$,

(d) *jeżeli* $\delta > 0$ *i* $E \in T(x)$, *to* $(x - \delta, x) \cap E$ *oraz* $(x, x + \delta) \cap E$ *są zbiorami niepustymi.*

Zbiory z rodziny $T(x)$ będziemy nazywać T-otoczeniami punktu x.

Definicja. *Liczbę $g \in \mathbb{R}$ nazywamy T-granicą funkcji f w punkcie x_0, jeżeli dla dowolnej liczby $\varepsilon > 0$ istnieje zbiór $E \in T(x_0)$ taki, że $|f(x) - g| < \varepsilon$ zachodzi dla wszystkich $x \in E$. Piszemy wówczas $T - \lim\limits_{x \to x_0} f(x) = g$.*

Oczywiście, funkcja nazywa się T-ciągła w x_0 jeżeli jest określona na pewnym T-otoczeniu x_0 oraz $T - \lim\limits_{x \to x_0} f(x) = f(x_0)$.

Analogicznie: T-pochodną funkcji f w punkcie x_0 nazywamy granicę $f'_T(x_0) = T - \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. Zauważmy, że jeżeli $T(x)$ jest rodziną przedziałów otwartych zawierających x, to T-granica oznacza zwykłą granicę. Jeżeli $T(x)$ jest rodziną wszystkich podzbiorów $\mathbb{R}$ zawierających x i takich, że ich gęstość w punkcie x jest równa 1, to pojęcie T-granicy pokrywa się z pojęciem granicy aproksymatywnej.

Obecnie podobne uogólnienia pochodnej istnieją pod nazwą „path derivatives" (zajmują się tym m.in. A. M. Bruckner, R. J. O'Malley, B. S. Thomson np. w [2]), zaś uogólnienia rodzin $T(x)$ jako systemy lokalne (local systems) ([14]).

Przy warunkach wystarczających monotoniczności funkcji rozważa się różne klasy funkcji, np. funkcje Darboux, funkcje aproksymatywnie ciągłe, itp. W jednej z prac Tadeusza Świątkowskiego opublikowanych w Zeszytach Naukowych Politechniki Łódzkiej ([6]) rozważane były funkcje $f\colon \mathbb{R} \to \mathbb{R}$ posiadające własności:

- ($\alpha 1$) f ma własność Darboux,
- ($\beta 1$) f jest pierwszej klasy Baire'a,
- ($\gamma 1$) dla dowolnych różnych $x_1, x_2 \in \mathbb{R}$ takich, że $f(x_1) \neq f(x_2)$ istnieje punkt $x_3 \in I(x_1, x_2)$ taki, że $f(x_3) \in I(f(x_1), f(x_2))$ i f jest ciągła w x_3,

($I(a, b)$ oznacza przedział o końcach a i b).

Wiadomo, że suma dwóch funkcji mających własność Darboux nie musi tej własności posiadać. Podobnie granica jednostajnie zbieżnego ciągu funkcji mających własność Darboux nie musi mieć własności Darboux. Zygmunt Zahorski pokazał ([17]), że jeżeli funkcja f należy do klasy J, gdzie $J = (\alpha 1) \cap (\beta 1)$, a funkcja g jest ciągła, to $f + g \in J$. Edward Kocela udowodnił ([4]), że granica jednostajnie zbieżnego ciągu funkcji klasy J również należy do J. Zatem przyjęcie dodatkowego założenia było niezbędne, by podane działania dawały w rezultacie funkcję tej samej klasy.

W pracy ([6]) rozważana jest klasa funkcji J^* spełniających wszystkie warunki ($\alpha 1$), ($\beta 1$), ($\gamma 1$). Okazuje się, że warunki te są niezależne (tzn.

($\alpha 1$) nie zależy od ($\beta 1$) i ($\gamma 1$), itd), w szczególności $J \neq J^*$. Autorzy udowodnili następujące twierdzenia:

Twierdzenie. *Jeżeli funkcja f spełnia warunki* ($\alpha 1$) *i* ($\gamma 1$)*, g jest ciągła, to $f+g$ spełnia warunek* ($\gamma 1$)*.*

Jeżeli funkcje f_n spełniają warunki ($\alpha 1$) *i* ($\gamma 1$)*,* ($n \in \mathbb{N}$) *i $f_n \to f$, to f spełnia warunek* ($\gamma 1$)*.*

Twierdzenie. *Jeżeli $f \in J^*$, g jest ciągła, to $f+g \in J^*$. Jeżeli $f_n \in J^*$, $n \in \mathbb{N}$ i $f_n \to f$, to $f \in J^*$.*

W 1982 roku w pracy [7] rozważano podobne klasy funkcji dla funkcji dwóch zmiennych $f\colon \mathbb{R}^2 \to \mathbb{R}$:

($\alpha 2$) f ma własność Darboux,
($\beta 2$) f jest pierwszej klasy Baire'a,
($\gamma 2$) dla dowolnych różnych punktów (x_1, y_1), (x_2, y_2) płaszczyzny takich, że $f(x_1, y_1) \neq f(x_2, y_2)$, istnieje punkt $(x_3, y_3) \in I\left((x_1, y_1), (x_2, y_2)\right)$ taki, że $f(x_3, y_3) \in I\left(f(x_1, y_1), f(x_2, y_2)\right)$ i f jest ciągła w (x_3, y_3).

i okazało się, że funkcje dwóch zmiennych zachowują się odmiennie niż funkcje jednej zmiennej:

Twierdzenie. *Jeżeli $f_n \in (\alpha 2) \cap (\beta 2)$, $n \in \mathbb{N}$ i $f_n \rightrightarrows f$, to $f \in (\alpha 2) \cap (\beta 2)$. Istnieje ciąg $f_n \in (\alpha 2) \cap (\gamma 2)$, $n \in \mathbb{N}$ zbieżny jednostajnie do funkcji f taki, że $f \notin (\gamma 2)$.*

W podanej pracy po raz pierwszy funkcje z klasy ($\gamma 1$) i ($\gamma 2$) nazwano funkcjami Świątkowskiego. Nazwa ta została wprowadzona przez profesora Władysława Wilczyńskiego oraz dr Helenę Nonas (obecnie Pawlak). W 1995 roku Aleksander Maliszewski zdefiniował silną funkcję Świątkowskiego[1]:

Definicja ([5])**.** *Mówimy, że $f\colon \mathbb{R} \to \mathbb{R}$ jest silną funkcją Świątkowskiego ($f \in \acute{S}_s$), jeżeli dla dowolnych liczb $x_1, x_2 \in \mathbb{R}$ takich, że $f(x_1) \neq f(x_2)$ i każdego $\lambda \in I(f(x_1), f(x_2))$ istnieje punkt $x_3 \in I(x_1, x_2) \cap C_f$ taki, że $f(x_3) = \lambda$ (gdzie C_f oznacza zbiór wszystkich punktów ciągłości funkcji f).*

Zaś w 2002 Paulina Szczuka wprowadziła pojęcie ekstra silnej funkcji Świątkowskiego[2]:

[1] W angielskiej terminologii, A. Maliszewski zastosował termin *Świątkowski function*, a potem dla funkcji spełniającej silniejszy warunek: *strong Świątkowski function.* Dla uporządkowania terminologii zgodnej z duchem języka angielskiego należałoby napisać *function fulfilling strong condition of Świątkowski*

[2] Taka sama uwaga dotyczy ekstra silnej własności Świątkowskiego. Powinno się napisać: *function fulfilling extra strong condition of Świątkowski.*

Definicja ([8]). *Mówimy, że* $f : \mathbb{R} \to \mathbb{R}$ *spełnia ekstra silny warunek Świątkowskiego (jest ekstra silną funkcją Świątkowskiego), jeżeli dla dowolnych* $x_1, x_2 \in \mathbb{R}$ *takich, że* $x_1 < x_2$*, spełniona jest równość oraz*

$$f([x_1, x_2]) = f([x_1, x_2] \cap C_f).$$

Obecnie funkcje Świątkowskiego stały się podstawą wielu badań, publikacji, prac doktorskich.

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